QUADRATIC EQUATION

* Algebraic expression

$$P(x)= a_nx^{n} + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x^1 + a_0x^0$$

(i) $a_n \neq 0$ (ii) Power of x is whole number

is called polynomial.

eg: $x^3-7x^2+\pi x+5$ ✓ , $x^7-6x^5+ex^{-3}+7$ ✗, $ex^3-\pi x^2+\sin x$ ✗

* $a_n \rightarrow$ leading coefficient * $a_nx^n \rightarrow$ leading term.

if leading coefficient is ~~called~~ '1' the polynomial is called as monic polynomial.

eg: $5x^6-7x+8$ → Monic polynomial ✗

$5x^2-6x+3x^2+x^4$ → Monic polynomial ✓

→ Highest power of x in polynomial

Degree	Name	General Form	Example
(undefined)	Zero polynomial	$0 \cdot x^{2,3,5,\dots}$	0
0	(Non-zero) constant polynomial	$a; (a\neq 0)$	$-2, -1, 1, 7, 3$
1	Linear polynomial	$ax+b; (a\neq 0)$	$x+1$
2	Quadratic polynomial	$ax^2+bx+c; (a\neq 0)$	x^2+1
3	Cubic polynomial	$ax^3+bx^2+cx+d; (a\neq 0)$	x^3+1
4	Bi-Quadratic	$ax^4+bx^3+cx^2+dx+e; (a\neq 0)$	x^4+x^3+1

*Quad. Eqn and its solution :-

$$ax^2+bx+c=0,\ (a\neq 0)$$

$$\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$\left(\frac{\text{coeff. of } x}{2}\right)^2$ ↳ add + Subtract

$$\Rightarrow x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$b^2-4ac=D=\text{Discriminant}$$

$$\Rightarrow x=\frac{-b\pm\sqrt{D}}{2a}\ \Bigg\}\quad \alpha=\frac{-b+\sqrt{D}}{2a},\ \beta=\frac{-b-\sqrt{D}}{2a}$$

"fundamental theorem of algebra"

⇕

No: of roots of a poly: of degree n is exactly n in number real or imaginary counted with multiplicity

$$(x-1)^3=0$$

↳ cubic

→ solution = 1 (one)

→ roots = 1, 1, 1 (three)

* Relation b/w roots

(1) Sum of roots

$$\alpha+\beta=\frac{-b+\sqrt{D}}{2a}+\frac{-b-\sqrt{D}}{2a}=\frac{-2b}{2a}=\frac{-b}{a}$$

$$\Rightarrow S=\alpha+\beta=\frac{-b}{a}$$

(2) Products of roots

$$P=\alpha\cdot\beta=\left(\frac{-b+\sqrt{D}}{2a}\right)\left(\frac{-b-\sqrt{D}}{2a}\right)$$

$$=\frac{b^2-D}{4a^2}=\frac{\cancel{b^2}-\cancel{b^2}+4ac}{4a^2}$$

$$\Rightarrow P=\alpha\cdot\beta=\frac{c}{a}$$

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*Factorizing a Quadratic $P(x) = ax^2+bx+c \begin{cases}\to \alpha \\ \to \beta\end{cases}$ → Roots

$$P(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left(x^2 - (\alpha+\beta)x + \alpha\beta\right)$$

$$= a\left(x^2 - \alpha x - \beta x + \alpha\beta\right)$$

$$= a\left(x(x-\alpha) - \beta(x-\alpha)\right)$$

$$= a(x-\alpha)(x-\beta)$$

* Nature of roots

* if $D>0 \Rightarrow$ roots are real and distinct

$$\left\{ x = \frac{-b\pm\sqrt{D}}{2a} \begin{cases} \to \alpha = \frac{-b+\sqrt{D}}{2a} \\ \to \beta = \frac{-b-\sqrt{D}}{2a} \end{cases} \right\}$$

* if $D=0 \Rightarrow$ roots are real and equal

$$\left\{ \therefore \alpha = \beta = \frac{-b}{2a} \right\}$$

* if $D<0 \Rightarrow$ Imaginary roots

Note: for real roots $D \geq 0$

* Some important points

P(1): if $a,b,c \in \mathbb{Q}$ and D is perfect square then roots are rational.

Ex: $x^2 - 2\sqrt{3}x - 1 = 0$ [if $a,b,c \notin \mathbb{Q}$]

$$x = \frac{2\sqrt{3} \pm \sqrt{16}}{2} = \sqrt{3} \pm 2$$

↳ Not Rational

* x^2+x+1

$$= (x-\omega)(x-\omega^2)$$

* $x^2+1=0$

$$x^2 = -1$$

$$x = \pm i$$

$$\therefore x^2+1 = (x+i)(x-i)$$

$$(\alpha-\beta)^2 = \alpha^2+\beta^2-2\alpha\beta$$

$$= (\alpha+\beta)^2 - 4\alpha\beta$$

$$= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a}$$

$$= \frac{b^2}{a^2} - \frac{4c}{a}$$

$$(\alpha-\beta)^2 = \frac{b^2-4ac}{a^2}$$

$$(\alpha-\beta)^2 = \frac{D}{a^2}$$ [Always Applicable]

$$|\alpha-\beta| = \frac{\sqrt{D}}{|a|}$$ [Not valid]

$x^3+1=0$ — $D=-4$

$a=1$

$\alpha = i \quad \beta = -i$

$$|i-(-i)|$$

$$= |2i| = 2 \neq \frac{\sqrt{-4}}{1}$$

P(2): $a=1$, $b, c \in I$, D is square of an integer ~~Does~~ then roots are integers

(even + even = even
odd − even = odd)

Proof: $x^2+bx+c=0$, $b, c \in I$, D is square of an integer

$$x = \frac{-b \pm \sqrt{D}}{2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

→ b = even integer ⇒ $\frac{\overset{\text{even}}{-b} \pm \overset{\text{even}}{\sqrt{b^2-4c}}}{\underset{\text{even}}{2}} =$ integer

→ b = odd integer $\frac{\overset{\text{odd}}{-b} \pm \boxed{\sqrt{b^2-4c}}}{\underset{\text{even}}{2}} = \text{integer}$ → odd (odd / even)

(odd − odd = even)

P(3): $a, b, c \in Q$, D is not perfect square but $D > 0$ then roots are irrational and occurs in conjugate pair of surds. i.e form of $p+\sqrt{q}$, $p-\sqrt{q}$

$$x = \frac{-b+\sqrt{D}}{2a}, \frac{-b-\sqrt{D}}{2a}$$

$$= \frac{-b}{2a} + \frac{\sqrt{D}}{2a}, \frac{-b}{2a} - \frac{\sqrt{D}}{2a}$$

surds
↳ √ो perfect Square नहीं ~~होते~~ होते.
conjugate
↳ जिस Number से multiply करने पर Rational no: आजाएं

P(4): $a, b, c \in R$, $D < 0$
then ~~roo~~ imaginary roots and occurs in conjugate pairs. i.e one roots is $p+iq$ then other root is $p-iq$

Note: $a, b, c \in R$ → one root is $a+ib$ then 2nd root is $a-ib$

if $a, b, c \in Q$ → one root of quadratic is $3+\sqrt{7}$ then other root is $3-\sqrt{7}$

$x^2+x+1=0$

$$x = \frac{-1 \pm \sqrt{3}i}{2}$$

$$x = \underbrace{\frac{-1+\sqrt{3}i}{2}}_{\omega}, \underbrace{\frac{-1-\sqrt{3}i}{2}}_{\omega^2}$$

P(5): If $a+b+c=0$ then roots of quad. $ax^2+bx+c=0$ are $1, \frac{c}{a}$.

Proof $a(1)^2+b(1)+c=0 \Rightarrow$ is root:

$$\therefore ax^2+bx+c=0 \begin{matrix}\nearrow 1\\ \searrow \beta\end{matrix}$$

$$\therefore P=1\cdot\beta=\frac{c}{a} \Rightarrow \left(\beta=\frac{c}{a}\right)$$

P(6): if $a-b+c=0$ then roots of quad. $ax^2+bx+c=0$ are $-1, -\frac{c}{a}$

Proof $\therefore ax^2+bx+c=0 \begin{matrix}\nearrow -1\\ \rightarrow \beta\end{matrix}$

$$\therefore P=(-1)(\beta)=\frac{c}{a} \Rightarrow \left(\beta=-c/a\right)$$

* Equation from roots:

roots $=\alpha, \beta$

$$\therefore eq^n = (x-\alpha)(x-\beta)=0$$
$$\Rightarrow x^2-\beta x-\alpha x+\alpha\beta=0$$
$$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$$
$$\Rightarrow x^2-Sx+P=0 \quad \begin{bmatrix} S=\alpha+\beta \\ P=\alpha\cdot\beta \end{bmatrix}$$

Q: If $p(q-r)x^2+q(r-p)x+r(p-q)$ has equal roots prove $\frac{2}{q}=\frac{1}{p}+\frac{1}{r}$

M-1 $D=0$ {roots equal}

$$D=(q(r-p))^2-4p(q-r)r(p-q)=0$$

$$\Rightarrow \underbrace{(-q(p-r))}_{A+B}{}^2-4\underbrace{p(q-r)}_{A}\cdot\underbrace{r(p-q)}_{B}=0$$

$A=p(q-r)=pq-pr$

$B=r(p-q)=rp-qr$

$A+B=pq-qr$

$A+B=q(p-r)$

$$\Rightarrow (A+B)^2-4AB=0$$
$$\Rightarrow A^2+B^2+2AB-4AB=0$$
$$\Rightarrow (A-B)^2=0 \Rightarrow \{A=B\}$$

$\therefore p(q-r)=r(p-q)$

$pq-pr=pr-qr$

$2pr=pq+qr$

$\overline{pqr}\left(\ \frac{2}{q}=\frac{1}{r}+\frac{1}{p}\right.$ Q.E.D

M-11 $\underbrace{p(q-r)}_{a}x^2+\underbrace{q(r-p)}_{b}x+\underbrace{r(p-q)}_{c}=0$

$$\Rightarrow a+b+c=\cancel{p}q-pr+qr-qp+rp-rq=0$$

$$\Downarrow$$

roots are $1, \frac{c}{a}$

we given roots are equal

$$1=\frac{c}{a}\Rightarrow c=a\Rightarrow r(p-q)=p(q-r)$$

$$pr-qr=pq-pr$$

$$2pr=pq+qr$$

$$\overline{pqr}\quad\left\{\frac{2}{q}=\frac{1}{r}+\frac{1}{p}\right\}$$

Q: If α,β are roots of $x^2-2x+5=0$ then form a quadratic equation whose roots are

$\alpha^3+\alpha^2-\alpha+22$ & $\beta^3+4\beta^2-4\beta+35$

to form a quadratic whose roots are $\alpha^3+\alpha^2-\alpha+22$ & $\beta^3+4\beta^2-4\beta+35$

$$x^2-2x+5=0 \begin{matrix}\nearrow \alpha\\ \searrow \beta\end{matrix}$$

$$\left.\begin{matrix}\alpha^2-2\alpha+5=0\\ \beta^2+2\beta+5=0\end{matrix}\right]$$

जब Higher degree से Lower degree में जाते हैं तो Division method से!!

सोच

$$\overset{\alpha^3}{p(x)}=\underbrace{q(x)}_{0}\cdot g(x)+r(x)$$

$$=\underline{r(x)}$$

$$\begin{array}{r|l} & \alpha+3\\ \alpha^2-2\alpha+5 & \alpha^3+\alpha^2-\alpha+22\\ & \underline{\alpha^3-2\alpha^2+5\alpha}\\ & 3\alpha^2-6\alpha+22\\ & \underline{3\alpha^2-6\alpha+15}\\ & 7 \end{array}$$

$$\therefore \alpha^3+\alpha^2-\alpha+22=(\alpha^2-2\alpha+5)(\alpha+3)$$

$$\alpha^3+\alpha^2-\alpha+22=+7\ (\text{one root})\ +7$$

$$\begin{array}{r|l|l} \beta^2+2\beta+5 & \beta^3+4\beta^2-7\beta+35 & \beta\\ & \underline{\beta^3-2\beta^2+5\beta} & \\ & 6\beta^2-12\beta+35 & \\ & \underline{6\beta^2-12\beta+30} & \\ & 5 & \end{array}$$

$$\therefore \beta^3-4\beta^2-7\beta+35$$
$$=(\beta^2+2\beta+5)(\beta+6)$$

$$\rightarrow \beta^3-4\beta^2-7\beta+35=+5\ (\text{2nd root})\ +5$$

$\therefore$ Eqⁿ with roots 5 & 7

$$\therefore\ \underline{x^2-12x+35=0}$$

* Symmetric function of roots:

if $f(\alpha,\beta) = F(\beta,\alpha)$ then $f(\alpha,\beta)$ is symmetric func^n of roots

eg: $f(\alpha,\beta) = \alpha^2+\beta^2,\ \alpha^2\beta+\beta^2\alpha+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$f(\beta,\alpha) = \beta^2+\alpha^2+\beta^2\alpha+\alpha^2\beta+\frac{\beta}{\alpha}+\frac{\alpha}{\beta}$

Note: Every symmetric function in α,β can be expressed in terms of two symmtric function.

$\alpha+\beta$ and $\alpha\cdot\beta$

NICHOD! Value of symmetric Expression in α,β can be found using sum of roots & Product of roots.

* A Golden Point:

(i) if $D_1 + D_2 \geq 0 \Rightarrow$
- Atleast one equation has real roots.
- if roots of one equation are imaginary then those of other equation will be real & unequal

(ii) if $D_1 + D_2 < 0 \Rightarrow$
- Atleast one of the equation has imaginary roots
- if roots of one equation are real then the other equation will be imaginary.

* Quadratic Equation v/s Identity:

If a quadratic has more than 2 distinct roots then it becomes an identity

eg) $(a+b)^2 = a^2+b^2+2ab$ (Idnty) $\quad \sec^2\theta - \tan^2\theta = 1 \quad \theta \neq (2n+1)\frac{\pi}{2}$ (Identity)

$2a+3=5$ (Eq^n)

Proof: $ax^2+bx+c=0 \to \alpha, \beta, \gamma$ } $\to$ distinct

$$\left.\begin{aligned} a\alpha^2+b\alpha+c&=0\\ a\beta^2+b\beta+c&=0\\ a\gamma^2+a\gamma+c&=0\end{aligned}\right\}$$

$\to a(\alpha^2-\beta^2)+b(\alpha-\beta)=0 \Rightarrow a(\alpha+\beta)+b=0$

$\to a(\beta^2-\gamma^2)+b(\beta-\gamma)=0 \Rightarrow a(\beta+\gamma)+b=0$

$a(\alpha-1)=0 \Rightarrow a=0$

$b=0$

$c=0$

Now. Quad. become

$0\cdot x^2+0\cdot x+0=0$ (identity).

NOTE :

- * In any polynomial equation, if the number of roots > degree of equation then it becomes an identity.
- * In any polynomial equation becomes an identity, then its all coefficients are simultaneously zero
  ↳ eg:) $\{0\cdot x^3 - 0\cdot x^2 + 0\cdot x + 0 = 0\}$

*** Newton's Formula:→**

$ax^2+bx+c=0 \begin{matrix}\nearrow \alpha \\ \searrow \beta\end{matrix}$, $S_n=\alpha^n+\beta^n$, $T_n=\alpha^n-\beta^n$

$a\alpha^2+b\alpha+c=0 \times \alpha^{n-2} \Rightarrow a\alpha^n+b\alpha^{n-1}+c\alpha^{n-2}=0$

$a\beta^2+b\beta+c=0 \times \beta^{n-2} \Rightarrow a\beta^n+b\beta^{n-1}+c\beta^{n-2}=c$

Add $a(\alpha^n+\beta^n)+b(\alpha^{n-1}+\beta^{n-1})+c(\alpha^{n-2}+\beta^{n-2})=0$

$$\Rightarrow \boxed{aS_n+bS_{n-1}+c\cdot S_{n-2}=0}$$

subtract: $a(\alpha^n-\beta^n)+b(\alpha^{n-1}-\beta^{n-1})+c(\alpha^{n-2}-\beta^{n-2})=0$

$$\boxed{aT_n+bT_{n-1}+cT_{n-2}=0}$$

Q $ax^2+bx+c=0 \begin{matrix}\nearrow \alpha \\ \searrow \beta\end{matrix}$, find $\alpha^4+\beta^4=$??

$aS_n+bS_{n-1}+cS_{n-2}=0$

$n=4$ $aS_4+bS_3+cS_2=0$ —(1)

$n=3$ $aS_3+bS_2+cS_1=0$

$S_1=\alpha+\beta$

$S_2=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$\Downarrow$

S_3 → from 1st eqn we get S_4.

Q $x^2-6x-2=0 \begin{matrix}\nearrow \alpha \\ \searrow \beta\end{matrix}$, $a_n=\alpha^n-\beta^n$ then find $\frac{a_{10}-2a_8}{2a_9}$

$x^2-6x-2=0$

$\Rightarrow a_n-6a_{n-1}-2a_{n-2}=c$

$\boxed{n=10}$ $a_{10}-6a_9-2a_8=0$

$\Rightarrow a_{10}-2a_8=3\cdot 2a_9 \Rightarrow \frac{a_{10}-2a_8}{2a_9}=③$ Ans.

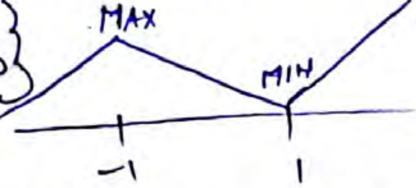
* MATHEMATICAL GYAAN:→

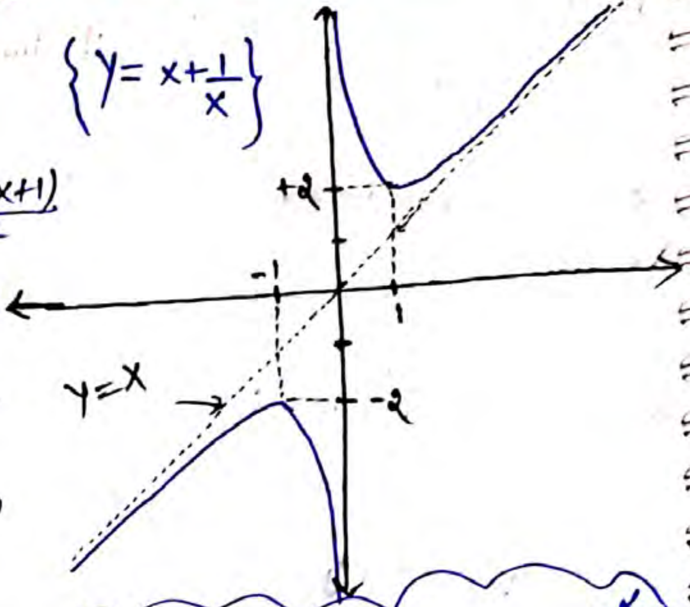
Graph: $y = x + \frac{1}{x}$

↳ $\frac{dy}{dx} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$

$\frac{1}{\text{very small +ve no.}} \to \infty$

$\frac{1}{\text{very small -ve no.}} \to -\infty$





$\lim_{x\to 0^+} x+\frac{1}{x} = 0+\infty = \infty$

$\lim_{x\to 0^-} x+\frac{1}{x} = -\infty$

$a(x-\alpha)(x-\beta) = ax^2+bx+c \rightarrow \alpha_1, \alpha_2$

$\alpha_1+\alpha_2 = -b/a$, $\alpha_1\alpha_2 = \frac{c}{a}$

* General Polynomial Equation:→

$a_0x^3+a_1x^2+a_2x+a_3=0 \rightarrow \alpha_1, \alpha_2, \alpha_3$

$a_0x^3+a_1x^2+a_2x+a_3 = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$

~~$= a_0(x^3-x^2(\alpha_1+\alpha_2+\alpha_3)+x(\alpha_1\alpha_2+\alpha_2\alpha_3$~~

$= a_0(x^3-x^2(\alpha_1+\alpha_2+\alpha_3)+x(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)-\alpha_1\alpha_2\alpha_3)$

$= a_0x^3-a_0x^2(\alpha_1+\alpha_2+\alpha_3)+a_0x(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)-a_0\alpha_1\alpha_2\alpha_3$

coefficient of x^2 $a_1 = -a_0(\alpha_1+\alpha_2+\alpha_3)$

coefficient of x $a_2 = a_0(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)$

constant term $a_3 = -a_0\alpha_1\alpha_2\alpha_3$

∴ $S_1 = \alpha_1+\alpha_2+\alpha_3 = (-1)^1\frac{a_1}{a_0}$

$S_2 = \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = (-1)^2\frac{a_2}{a_0}$

$S_3 = \alpha_1\alpha_2\alpha_3 = (-1)^3\frac{a_3}{a_0}$

S_1, S_3, S_5 ---- sab minus ke saath aatey hai.

S_2, S_4, S_6 ----- yeh positive ke saath.

eg: $2x^3-7x^2+6x+5 \rightarrow \alpha, \beta, \gamma$

$S_1 = \alpha+\beta+\gamma = -(-7/2) = 7/2$

$S_2 = \alpha\beta+\beta\gamma+\gamma\alpha = 6/2 = 3$

$S_3 = \alpha_1\alpha_2\alpha_3 = -5/2.$

* $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \begin{cases} \nearrow \alpha_1 \\ \to \alpha_2 \\ \searrow \alpha_3 \\ \downarrow \alpha_4 \end{cases}$

$S_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\left(\frac{a_1}{a_0}\right)$

$S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = \left(\frac{a_2}{a_0}\right)$

$S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4 + \alpha_1 a_3\alpha_4 = -\left(\frac{a_3}{a_0}\right)$

$S_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{a_4}{a_0}$

∴ $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \dots + a_{n-1}x + a_n = 0 \begin{cases} \nearrow \alpha_1 \\ \nearrow \alpha_2 \\ \nearrow \alpha_3 \\ \vdots \\ \searrow \alpha_n \end{cases}$

$S_1 = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n = -\frac{a_1}{a_0}$

$S_2 =$ product taken two at a time $= \frac{a_2}{a_0}$

$\vdots$

$S_n = \alpha_1\alpha_2\alpha_3 \dots \alpha_n = (-1)^n\left(\frac{a_n}{a_0}\right)$

NOTE: Every odd degree polynomial equation with real coefficient must have at least one real root. Because imaginary roots occur in conjugate pair.

eg. $ax^3 + bx^2 + cx + d = 0 \begin{cases} \nearrow 3+2i \\ \to 5 \\ \searrow 7 \end{cases}$ ⊗ $ax^3 + bx^2 + cx + d = 0 \begin{cases} \nearrow 3+2i \\ \to 3-2i \\ \searrow 5 \end{cases}$ ✓

$(a, b, c, d \in R)$

eg Possible Cases for 5 degree eqn
- All 5 roots real ✓
- 4 real & 1 imaginary X
- 3 real & 2 imaginary ✓
- 2 real & 3 imaginary X
- 1 real & 4 imaginary ✓
- All 5 imaginary. X

* Transformation of Equation:

$x^2 - 5x + 6 = 0 \begin{cases} \nearrow 2 = \alpha \\ \searrow 3 = \beta \end{cases}$ form a quad. whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$

$\underbrace{\frac{1}{\alpha}, \frac{1}{\beta} \quad f(\alpha) = \frac{1}{\alpha} \quad f(\beta) = \frac{1}{\beta}}$

∴ $f(x) = \frac{1}{x}$ (सोना)

Put $y = f(x) = \frac{1}{x} \Rightarrow x = \frac{1}{y}$ put in original eqn.

∴ $\frac{1}{y^2} - \frac{5}{y} + 6 = 0$

$\Rightarrow 6y^2 - 5y + 1 = 0 \xrightarrow[y \Leftrightarrow x]{\text{Replace}} 6x^2 - 5x + 1$

* Solution of Equations in two variables:

$$\begin{aligned} a_1x + b_1y + c_1 &= 0 \times b_2 \\ \underset{-}{a_2x} \underset{-}{+} b_2y \underset{-}{+} c_2 &= 0 \times b_1 \end{aligned}$$

$$(a_1b_2 - b_1a_2)x + b_2c_1 - b_1c_2 = 0$$

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - b_1a_2} = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\begin{aligned} a_1x + b_1y + c_1 &= 0 \times a_2 \\ \underset{-}{a_2x} \underset{-}{+} b_2\cancel{y} \underset{-}{+} c_2 &= 0 \times a_1 \end{aligned}$$

$$(b_1a_2 - a_1b_2)y + a_2c_1 - a_1c_2 = 0$$

$$y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - b_1a_2} = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Finally if $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

Then

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

* condition for common roots.

$$\left.\begin{aligned} a_1x^2 + b_1x + c_1 &= 0 \\ a_2x^2 + b_2x + c_2 &= 0 \end{aligned}\right\} \to \text{Have a common root } \alpha$$

$$\therefore \left.\begin{aligned} a_1\alpha^2 + b_1\alpha + c_1 &= 0 \\ a_2\alpha^2 + b_2\alpha + c_2 &= 0 \end{aligned}\right\} \alpha^2 = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad \alpha = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\therefore \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^{\cancel{2}}} = \frac{\begin{vmatrix} b & c_1 \\ b_2 & c_2 \end{vmatrix}}{\cancel{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}} \Rightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2$$

Finally

if $a_1x^2 + b_1x + c_1 = 0$, $a_2x^2 + b_2x + c_2 = 0$ have a common root α

then

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2$$

↳ This is condition for atleast one root common root.

2) **If Bothroots common.**

$$\therefore\ a_1x^2+b_1x+c_1=0 \begin{smallmatrix}\nearrow \alpha\\ \searrow \beta\end{smallmatrix} \rightarrow S=\alpha+\beta=-\frac{b_1}{a_1} \qquad P=\alpha\cdot\beta=\frac{c_1}{a_1}$$

$$a_2x^2+b_2x+c_2=0 \begin{smallmatrix}\nearrow \alpha\\ \searrow \beta\end{smallmatrix} \rightarrow S=\alpha+\beta=\frac{-b_2}{a_2} \qquad P=\alpha\cdot\beta=\frac{c_2}{a_2}$$

Hence condition for Both roots common,

$$\Downarrow$$

$$+\frac{b_1}{a_1}=+\frac{b_2}{a_2} \Big\} \quad \frac{c_1}{a_1}=\frac{c_2}{a_2}$$

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2} \qquad \Rightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2} \Big\} \quad \frac{a_1}{a_2}=\frac{c_1}{c_2}$$

NOTE:

$$\# \; a_1x^2+b_1x+c_1=0 \begin{smallmatrix}\nearrow \boxed{p+iq}\\ \searrow p-iq\end{smallmatrix}$$

$$a_2x^2+b_2x+c_2=0 \begin{smallmatrix}\nearrow \boxed{p+iq}\\ \searrow p-iq\end{smallmatrix}$$

Have a common roots & the roots of one of the equations are imaginary then both roots will be common

\# if the equation

$f(x)=0 \rightarrow \alpha$ } Have a common root say α then $x=\alpha$ is
$g(x)=0 \rightarrow \alpha$ } also $Af(x) \pm Bg(x)=0 \rightarrow \alpha$

eg: $x^2-5x+6=0 \begin{smallmatrix}\nearrow \boxed{2}\\ \searrow 3\end{smallmatrix}$

$\quad x^2-6x+8=0 \begin{smallmatrix}\nearrow 4\\ \searrow \boxed{2}\end{smallmatrix}$

$\quad \underline{-\quad +\quad -}$

$\quad x-2=0 \Rightarrow \boxed{x=2}$

Q ~~let α,β~~, $x^3+2x^2-4x+5=0 \begin{smallmatrix}\nearrow \alpha\\ \rightarrow \beta\\ \searrow \gamma\end{smallmatrix}$. find $\frac{(\alpha^3+5)(\beta^3+5)(\gamma^3+5)}{13\alpha\cdot\beta\cdot\gamma}$

$\alpha^3+5=4\alpha-2\alpha^2=2\alpha(2-\alpha)$

$\alpha^3+5=2\alpha(2-\alpha)$

lly $\beta^3+5=2\beta(2-\beta)$

$\gamma^3+5=2\gamma(2-\gamma)$

$$E=\frac{(\alpha^3+5)(\beta^3+5)(\gamma^3+5)}{13\alpha\cdot\beta\cdot\gamma}$$

$$=\frac{2\cancel{\alpha}(2-\alpha)\cdot 2\cancel{\beta}(2-\beta)\cdot 2\cancel{\gamma}(2-\gamma)}{13\cancel{\alpha}\cdot\cancel{\beta}\cdot\cancel{\gamma}}$$

put eqn (1)

$$\Rightarrow \frac{8(2-\alpha)(2-\beta)(2-\gamma)}{13}$$

$$\Rightarrow \frac{8\times 13}{13}=\boxed{8}$$

MAST TARIKA

$x^3+2x^2-4x+5=1\cdot(x-\alpha)(x-\beta)(x-\gamma)$

$\alpha \quad \beta \quad \gamma$

put $x=2$

$8+8-8+5=(2-\alpha)(2-\beta)(2-\gamma)$

$\Rightarrow (2-\alpha)(2-\beta)(2-\gamma)=13$ — (1)

* An Important Point : (Adv)

$$f(x) = ax^2+bx+c = \cancel{0}\ a(x-\alpha)(x-\beta)$$

$$\text{if } D=0 \Rightarrow \boxed{\alpha=\beta}$$

$$\therefore f(x) = a(x-\alpha)(x-\alpha)$$

$$f(x) = a(x-\alpha)^2$$

a = +ve

$$F(x) = (\sqrt{a})^2(x-\alpha)^2$$

$$= [\sqrt{a}\,(x-\alpha)]^2$$

⇓

f(x) becomes square of a linear polynomial with real coeff.

a = −ve

$$f(x) = (\sqrt{|a|}\, i)^2 (x-\alpha)^2$$

$$= [(\sqrt{|a|}\, i)(x-\alpha)]^2$$

$$(\sqrt{|-4|}\ i)^2$$

$$-4 = (2i)^2$$

$$-16 = (4i)^2$$

⇓

$$(\sqrt{|-16|}\ i)^2$$

⇓

$$(4i)^2$$

Q $F(x) = \lambda x^2 - 2\lambda x + 1$

Find λ if (a) f(x) is square of a linear/perfect square

sol^n $D=0$

$$4\lambda^2 - 4\lambda = 0 \ \& \ a>0 \quad \therefore \lambda > 0$$

$$4\lambda(\lambda-1) = 0$$

$$\underset{\otimes}{\lambda = 0},\ \lambda = 1$$

$f(x) = \lambda x^2 - \lambda x - 4$

Find λ if f(x) is square of linear with real coeff.

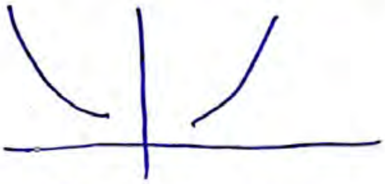
sol^n $D=0$

$$\lambda^2 + 16\lambda = \lambda(\lambda+16) = 0$$

$$\lambda = \underset{\otimes}{0},\ \underset{\otimes}{-16}$$

ANS: No sol^n value of λ exist!

* Analysis of a Quadratic Polynomial:



$$y = f(x) = ax^2+bx+c,\ (a \neq 0)$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

$$= a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

$$f(x) = y = a\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a}$$

$$f\left(\frac{-b}{2a}+k\right) = a\left(\frac{-b}{2a}+k+\frac{b}{2a}\right)^2 - \frac{D}{4a} = ak^2 - \frac{D}{4a}$$

$$f\left(\frac{-b}{2a}-k\right) = a\left(\frac{-b}{2a}-k+\frac{b}{2a}\right)^2 - \frac{D}{4a} = ak^2 - \frac{D}{4a}$$

Same.

$\rightarrow$ Graph of $y = f(x) = ax^n + bx + c$ is symmetric about $x = -b/2a$

$$y = f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

if $a > 0$

$y_{MAX} \rightarrow \infty$

$y_{MIN} \rightarrow \frac{-D}{4a}$ at $x = \frac{-b}{2a}$

if $a < 0$

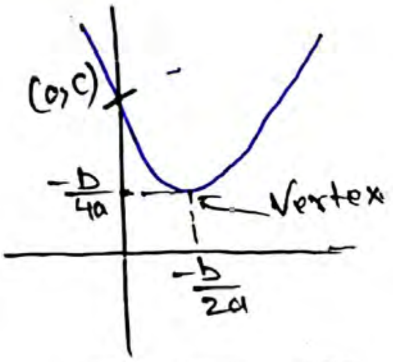
$y_{MIN} \rightarrow -\infty$

$y_{MAX} = \frac{-D}{4a}$ at $\frac{-b}{2a}$.

$a > 0$

$y = ax^2 + bx + c$

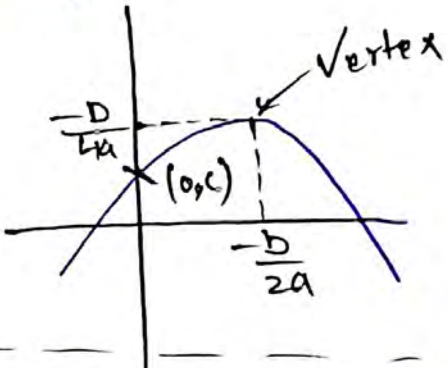
$y_{max} \rightarrow \infty$ $\quad$ $y_{min} = \frac{-D}{4a}$ at $\frac{-b}{2a}$.



$a < 0$

$y = ax^2 + bx + c$

$y_{max} = \frac{-D}{4a}$ at $x = \frac{-b}{2a}$ $\quad$ $y_{min} \rightarrow -\infty$

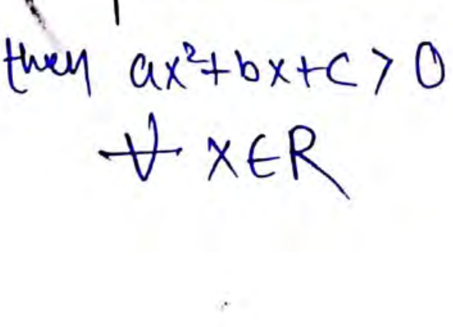


If $a > 0$ but $D < 0$

+ve

$y_{min} = \frac{-D}{4a}$

$y_{max} \rightarrow \infty$



then $ax^2 + bx + c > 0$

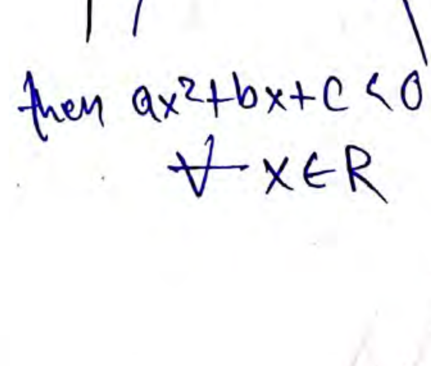
$\forall\ x \in R$

if $a < 0$ but $D < 0$

-ve

$y_{max} = \frac{-D}{4a}$

$y_{min} = \infty$



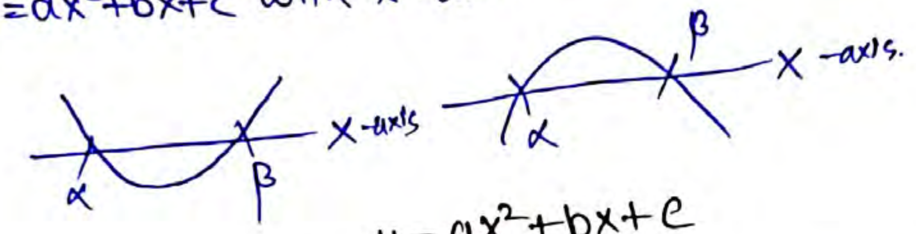
then $ax^2 + bx + c < 0$

$\forall\ x \in R$

* Quadratic Equation:

$y = ax^2 + bx + c \quad (a \neq 0)$

if we put $y = 0$ we get $ax^2 + bx + c = 0$ called Quadratic equation and roots are the nothing but x-coordinates of the point of intersection of $y = ax^2 + bx + c$ with x-axis.



$y = ax^2 + bx + c$

if $D = 0$

$a > 0$

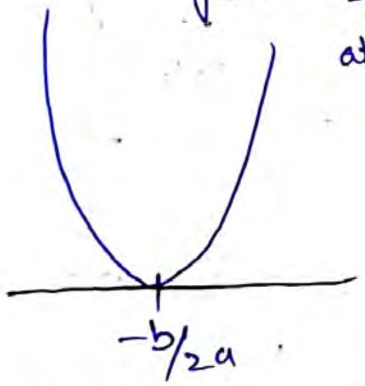
$y_{MAX} \to \infty$

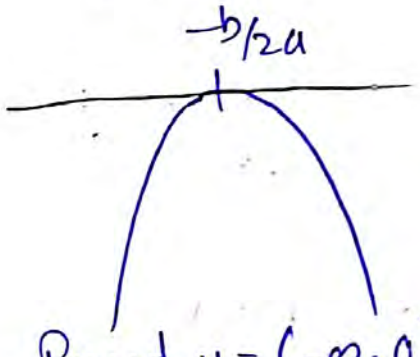
$y_{min} = \frac{-D}{4a} = 0$ at $x = \frac{-b}{2a}$

$a < 0$

$y_{MAX} = \frac{-D}{4a} = 0$ at $\frac{-b}{2a}$

$y_{MIN} \to -\infty$





Range of $y = (-\infty, 0]$

Range of $y = [0, \infty)$

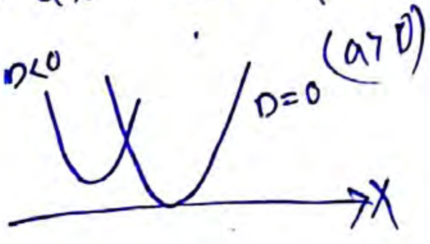
* Important:

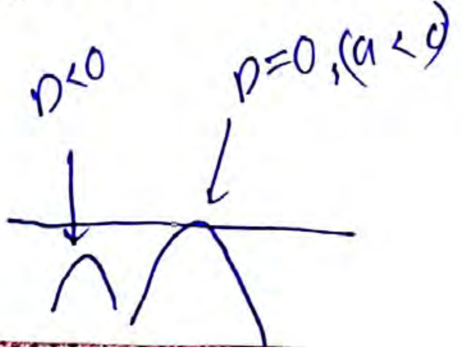
2, if $a > 0$ and $D < 0$ then $y = ax^2 + bx + C > 0 \ \forall \ x \in R$

2, if $a < 0$ and $D < 0$ then $y = ax^2 + bx + c < 0 \ \forall \ x \in R$

2, $ax^2 + bx + c \geq 0 \ \forall \ x \in R$ then $a > 0$ & $D \leq 0$

2, $ax^2 + bx + c \leq 0 \ \forall \ x \in R$ then $a < 0$ & $D \leq 0$





* MATHEMATICAL GYAAN :

Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in R$ Suppose $P(0)=6$, $P(1)=7$, $P(2)=8$ and $P(3)=9$, then find the value of $P(4)$.

Pattern को 4

let $g(x) = P(x) - (x+6) \rightarrow$ polynomial of degree 4

Pattern को पहचान कर Equation form करो!!

$$\left.\begin{aligned} g(0) &= P(0)-6=0 \\ g(1) &= P(1)-7=0 \\ g(2) &= P(2)-8=0 \\ g(3) &= P(3)-9=0 \end{aligned}\right\} \Rightarrow g(0)=g(1)=g(2)=g(3)$$

$$\Downarrow$$

$$g(x) = (x-0)(x-1)(x-2)(x-3)$$

$$P(x) - (x+6) = x(x-1)(x-2)(x-3)$$

$$\Rightarrow P(x) = x(x+1)(x-2)(x-3) + (x+6)$$

$$\Rightarrow P(4) = 4 \cdot 3 \cdot 2 \cdot 1 + 10 = \boxed{34}$$

* Range of Rational functions :-

Type(1): $f(x) = \dfrac{\text{linear}_1}{\text{linear}_2}$ i.e $f(x) = \dfrac{ax+b}{cx+d}$ $\left(\dfrac{a}{c} \neq \dfrac{b}{d}\right)$

if $\dfrac{a}{c} = \dfrac{b}{d}$

$f(x) = \dfrac{3x+2}{6x+4}$ $\left(\dfrac{3}{6} = \right.$

$= \dfrac{\cancel{3x+2}}{2(\cancel{3x+2})} = 1/2$

$$y = \frac{ax+b}{cx+d} \Rightarrow ycx + yd = ax + b$$

$$\Rightarrow x(cy-a) = b - yd$$

$$\Rightarrow x = \frac{b-yd}{cy-a} \Rightarrow cy - a \neq 0 \Rightarrow y \neq a/c$$

Hence $\Rightarrow y \in R - \left\{\dfrac{a}{c}\right\}$

eg. $f(x) = \dfrac{2x-3}{x-1} \Rightarrow R_f = R - \{2\}$

Type(2): $f(x) = \dfrac{\text{quad}_1}{\text{quad}_2}$ where numerator & denominator contain a common factor

Steps:

(i) Factorize numerator & Denominator

(ii) Say $(ax-b)$ is a common factor, cancel $(ax-b)$ from numerator & denominator & write $x = b/a$

(iii) Now $f(x) = \dfrac{px+q}{rx+s}$, $x \neq b/a$ & range $= R - \left\{\dfrac{p}{r}, f\left(\dfrac{b}{a}\right)\right\}$

eg: $f(x) = \dfrac{x^2-5x+6}{x^2-6x+8} \Rightarrow f(x) = \dfrac{(x-3)\cancel{(x-2)}}{\cancel{(x-2)}(x-4)}$ $(x \neq 2)$

$f(x) = \dfrac{x-3}{x-4}$, $(x \neq 2)$

Range: $R - \{1, 1/2\}$

if $x=2$, $f(x) = \dfrac{2-3}{2-4} = 1/2$

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* **Type(3)**: $f(x) = \frac{\text{Quad}}{\text{Quad}}, \frac{\text{linear}}{\text{Quad}}, \frac{\text{Quad}}{\text{linear}}$, where numerator & denominator contain no common factor.

Steps:

(i) Equate given expression to y

(ii) Cross multiply & make quad in x

(iii) Since $x \in R$, put $D \geq 0$ & get range of y

(iv) Equate coeff of $x^2 = 0$ to get $y = y_1$ put expression $= y_1$ & solve for x

If we get real x answer of step 3 is final

if we do not get real x exclude y_1 from answer.

Q Find values of 'a' for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

$$y = \frac{ax^2+3x-4}{3x-4x^2+a}$$

given: Range R, a = ?

$y \in R$

$$\Rightarrow y = \frac{ax^2+3x-4}{-4x^2+3x+a}$$

For range to be R: Numerator & Denominator should not have a common root

for common root
$$\begin{vmatrix} a & 3 \\ -4 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & -4 \\ 3 & a \end{vmatrix} = \begin{vmatrix} -4 & a \\ a & -4 \end{vmatrix}^2$$

$$\Rightarrow (3a+12)^2 = (16-a^2)^2$$

$$\Rightarrow (3a+12)^2 - (16-a)^2 = 0$$

$$\Rightarrow (3a+12+16-a^2)(3a+12-16+a^2) = 0$$

$$\Rightarrow (-a^2+3a+28)(a^2+3a-4) = 0$$

$$\Rightarrow (a^2-3a-28)(a^2+3a-4) = 0$$

$$\Rightarrow a^2-3a-28 = 0 \quad , \quad a^2+3a-4 = 0$$

$a = 7, -4$ $\qquad$ $a = -4, 1$

$$a \neq -4, 7, 1$$

$$y=\frac{ax^2+3x-4}{3x-4x^2+a}$$

$\Rightarrow 3xy-4yx^2+ay=ax^2+3x-4$

$\Rightarrow (a+4y)x^2+(3-3y)x-4-ay=0$

Since $x\in R$

$\underline{D\geqslant 0}$, $9(1-y)^2+4(a+4y)(4+ay)\geqslant 0$

$\Rightarrow 9y^2+9-18y+4(4a+a^2y+16y+4ay^2)\geqslant 0$

$\Rightarrow y^2(16a+9)+y(4a^2+46)+9+16a\geqslant 0 \;\forall\, y\in R$

$\overline{D\leqslant 0}$
$\underline{,16a+9>0}$ $(4a^2+46)^2-4\cdot(16a+9)(16a+9)\leqslant 0$, $\boxed{a>\frac{-9}{16}}$

$\Rightarrow (4a^2+46)^2-(2(16a+9)^2\leqslant 0$ —(i)

$\Rightarrow (4a^2+46-32a-18)(4a^2+46+32a+18)\leqslant 0$

$\Rightarrow (4a^2-32a+28)(4a^2+32a+64)\leqslant 0$

$\Rightarrow (a^2-8a+7)(a^2+8a+16)\leqslant 0$

$\Rightarrow (a-1)(a-7)(a+4)^2\leqslant 0$

$\Rightarrow (a-1)(a-7)\leqslant 0$, $a=-4$ is also possible.

$\Rightarrow a\in[1,7]\cup\{-4\}$ —(ii)

From eqⁿ (i) and (ii) $a\in[1,7]$ But $\boxed{a\neq 7,1}$

$\therefore$ ANS: $a\in(1,7)$

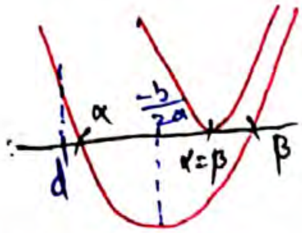
* **Location of roots** :→

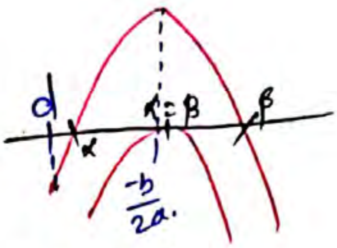
The article deals with an elegant approach of solving problems on quadratic equations when the roots are located/specified on the number line with variety of constraints: '

consider: $f(x)=ax^2+bx+c$

Type 1 : Both roots greater than no: 'd'

$f(x) = ax^2+bx+C$





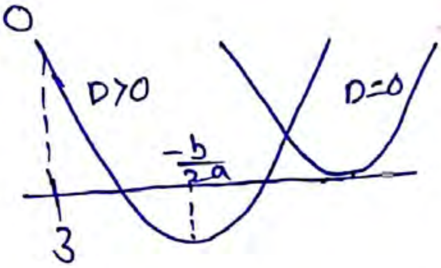
(1) $D \geq 0$	(1) $D \geq 0$
(2) $\frac{-b}{2a} > d$	(2) $\frac{-b}{2a} > d$
(3) $f(d) > 0$	(3) $f(d) < 0$
(4) $a > 0$	(4) $a < 0$

combine

(a) $D \geq 0$ (b) $\frac{-b}{2a} > d$ (c) $af(d) > 0$

eg: Find all the values of the parameter 'd' for which both the roots of the eqn $x^2 - 6dx + (2-2d-9d^2) = 0$ exceed the number 3

$f(x) = x^2 - 6dx + (2-2d-9d^2) = 0$



(1) $D \geq 0$

(2) $f(3) > 0$

(3) $\frac{-b}{2a} > 3$

(1) $D = 36d^2 - 4\cdot(2-2d+9d^2) \geq 0$

$\cancel{9d^2} - 2 + 2d - \cancel{9d^2} \geq 0$

$2d - 2 \geq 0$

$\boxed{d \geq 1}$ — I

(2) $f(3) > 0$

$9 - 18d + 2 - 2d + 9d^2 > 0$

$9d^2 - 20d + 11 > 0$

$9d^2 - 11d - 9d + 11 > 0$

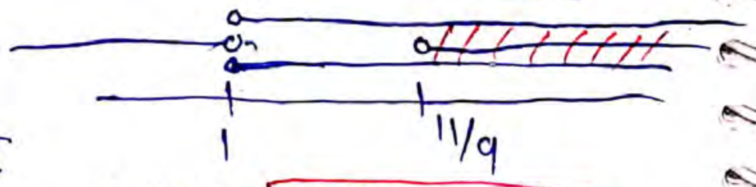
$(9d-11)(d-1) > 0$

$\boxed{d \in (-\infty, 1) \cup (11/9, \infty)}$ — II

(3) $\frac{-b}{2a} > 3$

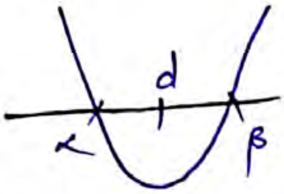
$\Rightarrow \frac{6d}{2} > 3 \Rightarrow 3d > 3$

$\Rightarrow \boxed{d > 1}$ — III

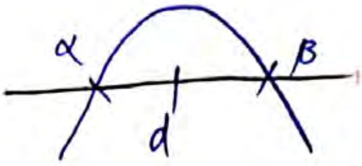


ANS: $\boxed{d \in (11/9, \infty)}$

Type 2: Both roots lie on either side of 'd' or d lies b/w roots or one root is less than 'd' & other root is greater than 'd'

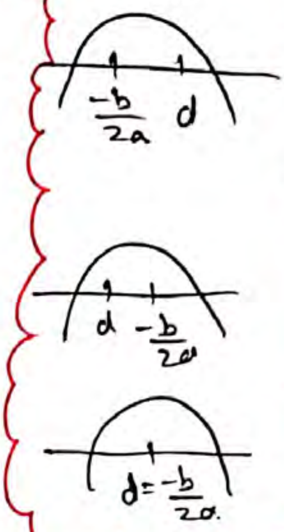


(i) $a > 0$
(ii) $D > 0$
(iii) $f(d) < 0$



(i) $a < 0$
(ii) $f(d) > 0$
(iii) $D > 0$

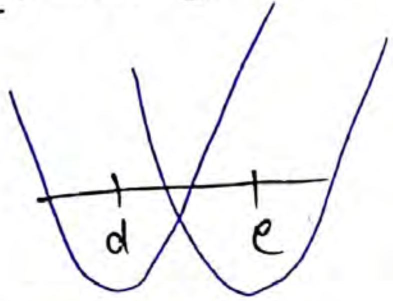
can not comment on $= \frac{-b}{2a}$



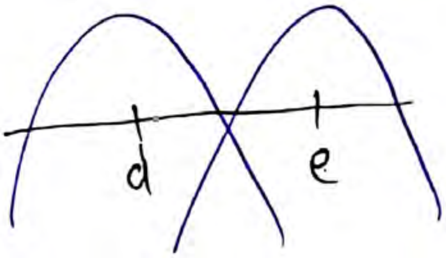
combine

(i) $a \cdot f(d) < 0$
(ii) $D > 0$ (No Need)

Type 3: Exactly one root lies b/w d & e $(d < e)$



(i) $f(d) \cdot f(e) < 0$
(ii) $a > 0$
(iii) $D > 0$



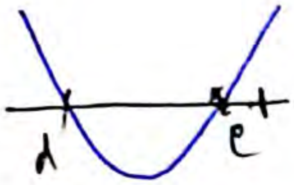
(i) $f(d) \cdot f(e) < 0$
(ii) $a < 0$
(iii) $D > 0$

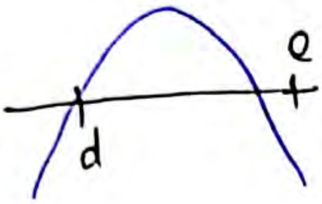
combine

(i) $f(d) \cdot f(e) < 0$
(ii) $a \neq 0$
(iii) $D > 0$ (No Need)

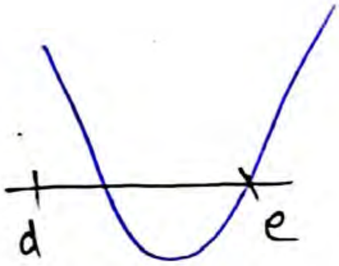
Next

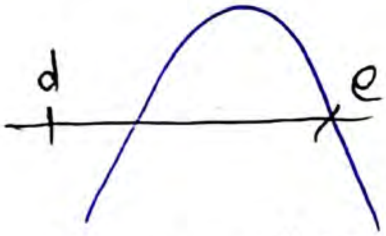
Now ~~two~~ possibilities Ariso :





One root lies at $x=d$ i.e $f(d)=0$, then find other root it must lie in (d,e)



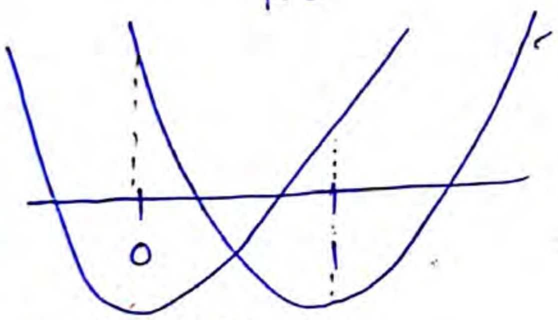


One root lies at $x=e$ i.e $f(e)=0$ then find the other root it must lie in (d,e)

Question : $(\lambda^2+1)x^2-4\lambda x+2=0$ always have exatcly one root in the interval $(0,1)$ is: find set of 'λ'

exaclly one root lies in $(0,1)$

$$y=\underbrace{(\lambda^2+1)}_{+ve}x^2-4\lambda x+2=0$$



$f(0)\cdot f(1)<0$

$\Rightarrow 2\cdot(\lambda^2+1-4\lambda+2)<0$

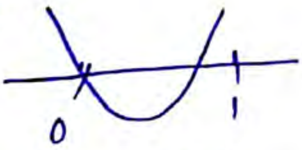
$\Rightarrow (\lambda^2-4\lambda+3)<0$

$\Rightarrow (\lambda-3)(\lambda-1)<0$

$\Rightarrow \lambda\in(1,3)$ — (i)

Now possibility arise

(P1)

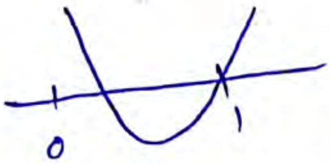


$f(0)=0 \Rightarrow 2=0$

↓

Not possible

(P2)



$f(1)=0 \Rightarrow \lambda^2+1-4\lambda+2=0$

$\Rightarrow \lambda^2-4\lambda+3=0$

$\underline{\lambda=1,3}$

If $\lambda = 1$ Eqⁿ becomes

$2x^2 - 4x + 2 = 0$

$x^2 - 2x + 1 = 0$

$x = 1, 1$

$\therefore \lambda = 1$ is rejected

If $\lambda = 3$ Eqⁿ becomes

$10x^2 - 12x + 2 = 0 \begin{smallmatrix} \nearrow 1 \\ \searrow \alpha \end{smallmatrix}$

M-1

$5x^2 - 6x + 1 = 0$

$(5x-1)(x-1) = 0$

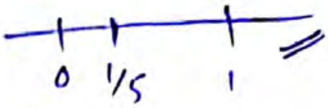
$x = 1/5, 1$

M-2 (easier method)

$\alpha \cdot 1 = 2/10$

$\boxed{x = 1/5}$

Hence $\lambda = 3$ is accepted



$\boxed{\therefore \ \lambda \in (1,3) \cup \{3\} \Rightarrow \lambda \in (1,3]}$

2. IIT can modify the question as!!

Q $x^2 - (a+1)x + 2a = 0$ has exactly one root in $[0,3]$ find range of a

Not $(0,3)$

$f(x) = x^2 - (a+1)x + 2a = 0$

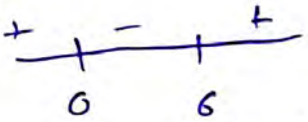
$f(0) \cdot f(3) < 0$

$\Downarrow$

$\Rightarrow 2a(a - 3a - 3 + 2a) < 0$

$\Rightarrow a(6-a) < 0$

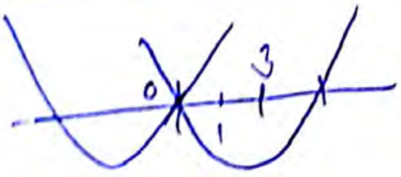
$\Rightarrow a(a-6) > 0$



$a \in (-\infty, 0) \cup (6, \infty)$ —(i)

Now two possibilities arise

Ⓟ₁



$f(0) = 0 \Rightarrow a = 0$

eqⁿ $x^2 - x = 0$ } $a \neq 0$ rejected ~~rejected~~

$x = 0, 1$

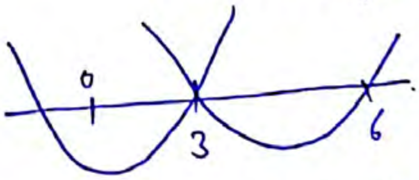
bcoz at $a = 0$

$x = 0, 1$ and $[0,3]$

~~has~~ x have 2 roots

but ACTQ only one root lie b/w $[0, 3]$

Ⓟ₂



$f(3) = 0 \Rightarrow 6 - a = 0 \Rightarrow \boxed{a = 6}$

eqⁿ $x^2 - 7x + 12 = 0 \begin{smallmatrix} \nearrow 3 \\ \searrow \beta \end{smallmatrix}$

$3 \cdot \beta = 12$

$\boxed{\beta = 4}$

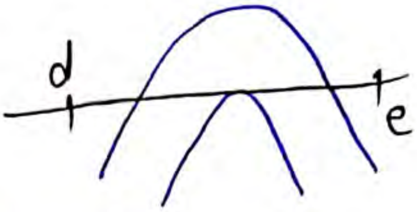
$(a = 6$ is accepted$)$ —(ii)

$\therefore \ a \in (-\infty, 0) \cup (6, \infty) \cup \{6\}$

$\Rightarrow \boxed{a \in (-\infty, 0) \cup [6, \infty)}$

Type IV : When both roots are confined b/w d & e (d<e)

OR



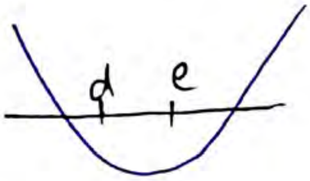
(1) $f(d) > 0$
(2) $f(e) > 0$
(3) $D \geq 0$
(4) $d < \frac{-b}{2a} < e$
(5) $a > 0$

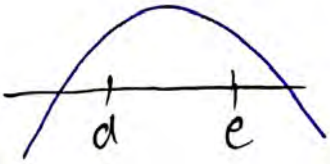
(1) $f(d) < 0$
(2) $f(e) < 0$
(3) $D \geq 0$
(4) $d < \frac{-b}{2a} < e$
(5) $a < 0$

combine

(i) $D \geq 0$ (ii) $d < \frac{-b}{2a} < e$
(iii) $a \cdot f(d) > 0$ (iv) $a \cdot f(e) > 0$

Type V : One root is greater than e & other root is less than d (d<e





(1) $a > 0$
(2) $f(d) < 0$
(3) $f(e) < 0$
(4) $D > 0$

(1) $a < 0$
(2) $f(d) > 0$
(3) $f(e) > 0$
(4) $D > 0$

combine

(i) $a \cdot f(d) < 0$
(ii) $a \cdot f(e) < 0$
(iii) $D > 0$ (No Need)

* **Integral / Rational Roots** : "All integers are rational"

$ax^2+bx+c=0$, where $a, b, c \in Q$ has rational roots if D is a perfect Square

Q Find no: of integral values α for which the quadratic eqn $x^2+\alpha x+\alpha+1=0$ has integral roots.

$x^2+\alpha x+\alpha+1=0$ we can assume α as integer clearly all coeff are rational

$\therefore$ for roots be rational $\Rightarrow$ D should be perfect square.

$$D=\alpha^2-4(\alpha+1)=\alpha^2-4\alpha-4=m^2, \; m \in N$$

$$\Rightarrow \alpha^2-4\alpha-4=m^2$$

$$\Rightarrow (\alpha-2)^2-8=m^2$$

$$\Rightarrow (\alpha-2)^2-m^2=8$$

$$\Rightarrow \overset{A}{(\alpha-2-m)}\overset{B}{(\alpha-2+m)}=8 \Rightarrow 1\times 8,\ 2\times 4,\ -8\times-1,\ -4\times-2$$

m plus होता है तो B A से बड़ा आएगा इसलिए sabhi cases की जरूरत नहीं

1×8 ✓ 8×1 ⊗

$$\begin{aligned} \alpha-2-m&=1 \\ \alpha-2+m&=8 \\ \hline 2\alpha-4&=9 \\ \alpha&=13/2 \end{aligned}$$

✗ (bcoz not an integer)

$$\begin{aligned} \alpha-2-m&=2 \\ \alpha-2+m&=4 \\ \hline 2\alpha-4&=6 \\ \alpha&=5 \end{aligned}$$

✓

$$\begin{aligned} \alpha-2-m&=-8 \\ \alpha-2+m&=-1 \\ \hline 2\alpha-4&=-9 \\ \alpha&=-5/2 \end{aligned}$$

✗ (bcoz not an integer)

$$\begin{aligned} \alpha-2-m&=-4 \\ \alpha-2+m&=-2 \\ \hline 2\alpha-4&=-6 \\ \alpha&=-1 \end{aligned}$$

✓

$\Rightarrow$ if $\alpha=-1$ or 5 root are rational

if $\alpha=-1$ eqn becomes $x^2-x=0 \Rightarrow x=0,1$

if $\alpha=5$ eqn becomes $x^2+5x+6=0 \Rightarrow x=-2,-3$

Both roots integer

Hence $\alpha=-1$ & $\alpha=5$ is valid for integral roots.

M-2 $\quad x^2+\alpha x+\alpha+1=0 \begin{matrix}\nearrow a\\ \searrow b\end{matrix}$ integral

$S = a+b = -\alpha$

$P = ab = \alpha+1$

$a+b+ab = 1$ $\quad$ { Simon's Factoring Technique }

$\Rightarrow (a+1)+b(a+1)-1 = 0$

$\Rightarrow (a+1)(b+1) = 2$

$\Rightarrow a+1=1$, $b+1=2$ $\quad$ OR $\quad$ $a+1=-2$, $b+1=-2$

$a=0, b=1$ $\quad$ $a=-2, b=-3$ $\quad$ $\boxed{\alpha=-1,5}$

$\Rightarrow -\alpha = a+b$ $\quad$ $-\alpha=-5$

$\boxed{\alpha=-1}$ $\quad$ $\therefore \boxed{\alpha=5}$

* condition for general two degree expression in x & y to be resolved as a product of two linear factors.

$$F(x,y) = ax^2+2hxy+by^2+2gx+2fy+c$$

$$= ax^2+(2hy+2g)x+by^2+2fy+c$$

{ let us Find its roots. }

$$\therefore x = \frac{-(2hy+2g)\pm\sqrt{(2hy+2g)^2-4\cdot a\cdot(by^2+2fy+c)}}{2a}$$

$$= \frac{-(hy+g)\pm\sqrt{(h^2-ab)y^2+(2hg-2af)y+g^2-ac}}{a}$$

$$x = \frac{-(hy+g)+\sqrt{(h^2-ab)y^2+(2hg-2af)y+g^2-ac}}{a} = \alpha$$

$$x = \frac{-(hy+g)-\sqrt{(h^2-ab)y^2+(2hg-2af)y+g^2-ac}}{a} = \beta$$

∴ factor of $ax^2+(2hy+2g)x+by^2+2fy+c = a(x-\alpha)(x-\beta)$

∴ for linear factors

$px+qy+r$
$\downarrow$
linear in x&y

$(h^2-ab)y^2 - y(2hg-2af) + g^2-ac$

should become a perfect square

$\Rightarrow \{D=0\}$

$$D=(2hg-2af)^2-4(h^2-ab)(g^2-ac)=0$$

$$\frac{(\ )}{4}\Big(\Rightarrow 4(hg-af)^2-(h^2-ab)(g^2-ac)=0$$

$$\Rightarrow h^2g^2+a^2f^2-2afhg-[h^2g^2-h^2ac-g^2ab+a^2bc]=0$$

$$\frac{(\ )}{a}\Big(af^2-2fgh+ch^2+bg^2-abc=0$$

$$\Rightarrow \boxed{abc+2fgh-af^2-bg^2-ch^2=0}$$

$f(x,y) = ax^2+2hxy+by^2+2gx+2fy+c$ can be resolved in to two linear factors if

$$\Delta = abc+2fgh-af^2-bg^2-ch^2=0$$

OR

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Q: $2x^2+3xy+y^2+3x+1+2y$ Factorize into linear factors

$$f(x,y) = 2x^2+3xy+y^2+3x+1+2y$$

$a=2$, $b=1$, $2h=3$, $2g=3$, $2f=2$, $c=1$

$h=3/2$, $g=3/2$, $f=1$

$$\Delta = abc+2fgh-af^2-bg^2-ch^2$$

$$= 2\cdot1\cdot1+2\cdot1\cdot\frac{3}{2}\cdot\frac{3}{2}-2\cdot1^2-1(3/2)^2-1\cdot(3/2)^2$$

$$= 2+9/2-2-9/4-9/4$$

$$= 0$$

∴ given Expression can be factorized into 2 linear factor.

$$f(x,y) = 2x^2 + 3x\cdot y + y^2 + 3x + 2y + 1$$

Steps to Factorize

* Factorize the Homogenous part (degree same)

$$2x^2 + 3xy + y^2 = 2x^2 + 2xy + xy + y^2$$
$$= 2x(x+y) + y(x+y)$$
$$= (2x+y)(x+y)$$

* Add λ to first factor & μ to second factor & put their product equal to given expression.

$$(2x+y+\lambda)(x+y+\mu) = 2x^2 + 3xy + y^2 + 3x + 2y + 1$$

coefficient of x : $2\mu + \lambda = 3$

coefficient of y : $\mu + \lambda = 2$

$\mu = 1, \lambda = 1$

$$\boxed{\therefore\ 2x^2 + 3xy + y^2 + 3x + 2y + 1 = (2x+y+1)(x+y+1)}$$

$2\mu x + \lambda x$

$\underbrace{(2\mu+\lambda)x}_{\text{coeff. of } x}$

$\mu y + \lambda y$

$\underbrace{(\mu+\lambda)y}_{\text{coeff of } y}$

* A Golden Point :

for $y = f(x) = ax^2 + bx + c \quad (a \neq 0)$

if $f(p) < 0$ and $f(q) > 0$

i.e $f(p)\cdot f(q) < 0 \Rightarrow$ then the equation $ax^2+bx+c=0$ has exactly one root lying b/w p & q.