

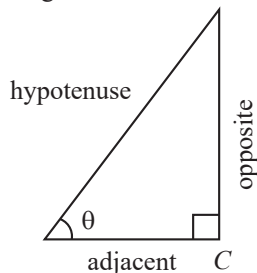
CHAPTER

4



Trigonometric Ratios and Identities

1. Trigonometric Ratio – Basic Terminology



The six trigonometric ratios of θ are defined as follows:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}, \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot \theta = \frac{\text{adjacent}}{\text{opposite}}, \operatorname{cosec} \theta = \frac{\text{hypotenuse}}{\text{opposite}}, \sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

2. Trigonometric identities:

- (i) $\sin^2 \theta + \cos^2 \theta = 1$
- (ii) $\sec^2 \theta - \tan^2 \theta = 1$
- (iii) $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$

3. Allied angles:

Two angles are said to be allied when their sum or difference is either zero or a multiple of $\frac{\pi}{2}$, two angles x, y are allied angles iff $|x \pm y| = 0$ or $\frac{n\pi}{2}, n \in N$.

$q \rightarrow$	$\frac{\pi}{2} - \theta$	$\frac{\pi}{2} + \theta$	$\pi - \theta$	$\pi + \theta$	$\frac{3\pi}{2} - \theta$	$\frac{3\pi}{2} + \theta$	$2\pi - \theta$	$2\pi + \theta$	$-\theta$
sin	$\cos \theta$	$\cos \theta$	$\sin \theta$	$-\sin \theta$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$	$\sin \theta$	$-\sin \theta$
cos	$\sin \theta$	$-\sin \theta$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$	$\sin \theta$	$\cos \theta$	$\cos \theta$	$\cos \theta$
tan	$\cot \theta$	$-\cot \theta$	$-\tan \theta$	$\tan \theta$	$\cot \theta$	$-\cot \theta$	$-\tan \theta$	$\tan \theta$	$-\tan \theta$
cot	$\tan \theta$	$-\tan \theta$	$-\cot \theta$	$\cot \theta$	$\tan \theta$	$-\tan \theta$	$-\cot \theta$	$\cot \theta$	$-\cot \theta$
sec	$\operatorname{cosec} \theta$	$-\operatorname{cosec} \theta$	$-\sec \theta$	$-\sec \theta$	$-\operatorname{cosec} \theta$	$\operatorname{cosec} \theta$	$\sec \theta$	$\sec \theta$	$\sec \theta$
cosec	$\sec \theta$	$\sec \theta$	$\operatorname{cosec} \theta$	$-\operatorname{cosec} \theta$	$-\sec \theta$	$-\sec \theta$	$-\operatorname{cosec} \theta$	$\operatorname{cosec} \theta$	$-\operatorname{cosec} \theta$

4. Sum & Difference Formula

- (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- (ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- (iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- (iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- (v) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- (vi) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
- (vii) $\cot(A + B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$
- (viii) $\cot(A - B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$

5. Product to sum

- (i) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$
- (ii) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$
- (iii) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$
- (iv) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

6. Trigonometric transformations:

- (i) $\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$
- (ii) $\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$
- (iii) $\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$

$$(iv) \cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

$$= 2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{D-C}{2}\right)$$

Some useful Formulae:

- (i) $\sin(A+B) \sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
- (ii) $\cos(A+B) \cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
- (iii) $\tan(A+B) \tan(A-B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$
- (iv) $\cot(A+B) \cot(A-B) = \frac{\cot^2 B \cot^2 A - 1}{\cot^2 B - \cot^2 A}$

7. Some standard trigonometric values

Angle → Trigonometric Function ↓	15°	18°	22½°	36°
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$
tan	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$

8. Double Angle / Triple Angle

- (i) $\sin 2A = 2 \sin A \cos A$
- (ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
Suppose that A is not an odd multiple of $\frac{\pi}{4}$. Then
- (iii) $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$
- (iv) $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$
- (v) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$ (Here $2A$ is also not an odd multiple of $\frac{\pi}{2}$)
- (vi) $\sin 3A = 3 \sin A - 4 \sin^3 A, \forall A \in R$
- (vii) $\cos 3A = 4 \cos^3 A - 3 \cos A, \forall A \in R$
- (viii) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$ ($3A, A$ are not odd multiples of $\frac{\pi}{2}$)
- (ix) $\cot 3A = \frac{3 \cot A - \cot^3 A}{1 - 3 \cot^2 A}$ ($3A, A$ are not multiples of $\frac{\pi}{2}$)

9. Half Angle

- (i) $\sin \frac{A}{2} = 2 \sin \frac{A}{4} \cos \frac{A}{4}$
- (ii) $\cos \frac{A}{2} = \cos^2 \frac{A}{4} - \sin^2 \frac{A}{4} = 2 \cos^2 \frac{A}{4} - 1 = 1 - 2 \sin^2 \frac{A}{4}$
- (iii) $\tan \frac{A}{2} = \frac{2 \tan \frac{A}{4}}{1 - \tan^2 \frac{A}{4}}$ (where $\frac{A}{2} \neq (2n+1)\frac{\pi}{2}, n \in Z$)

10. Conditional Identities

In a Triangle If $A + B + C = \pi$, then

- (i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- (ii) $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
- (iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- (iv) $\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + 4 \cos A \cos B \cos C)$
- (v) $\cos^2 A + \cos^2 B + \cos^2 C = (1 - 2 \cos A \cos B \cos C)$

11. Domains, Ranges and Periodicity of Trigonometric Functions:

T-Ratio	Domain	Range	Period
$\sin x$	R	$[-1, 1]$	2π
$\cos x$	R	$[-1, 1]$	2π
$\tan x$	$R - \left\{ (2n+1)\frac{\pi}{2}; n \in I \right\}$	R	π
$\cot x$	$R - \{n\pi; n \in I\}$	R	π
$\sec x$	$R - \left\{ (2n+1)\frac{\pi}{2}; n \in I \right\}$	$(-\infty, -1] \cup [1, \infty)$	2π
$\operatorname{cosec} x$	$R - \{n\pi; n \in I\}$	$(-\infty, -1] \cup [1, \infty)$	2π

12. Maxima-Minima

- (i) $a \cos \theta + b \sin \theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e., the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.
- (ii) Minimum value of $a^2 \tan^2 \theta + b^2 \cot^2 \theta$ is $2ab$ where $a, b > 0$.
- (iii) $-\sqrt{a^2+b^2+2ab \cos(\alpha-\beta)} < a \cos(\alpha+\theta) + b \cos(\beta+\theta) \leq \sqrt{a^2+b^2+2ab \cos(\alpha-\beta)}$ where α and β are known angles.



- (iv) If $\alpha, \beta, \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then
- (a) Maximum value of the expression $\cos \alpha \cos \beta, \cos \alpha + \cos \beta, \sin \alpha \sin \beta$ or $\sin \alpha + \sin \beta$ occurs when $\alpha = \beta = \frac{\sigma}{2}$
- (b) Minimum value of $\sec \alpha + \sec \beta, \tan \alpha + \tan \beta, \operatorname{cosec} \alpha + \operatorname{cosec} \beta$ occurs when $\alpha = \beta = \frac{\sigma}{2}$
- (v) If A, B, C are the angles of a triangle then maximum value of $\sin A + \sin B + \sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60^\circ$
- (vi) In case a quadratic in $\sin \theta$ & $\cos \theta$ is given then the maximum or minimum values can be obtained by making perfect square.
13. Sum of Three or More Angles
- (i) $\sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C = \sum \sin A \cos B \cos C - \prod \sin A$

$$(ii) \cos(A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \sin C = \prod \cos A - \sum \cos A \sin B \sin C$$

$$(iii) \tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{\sum \tan A - \prod \tan A}{1 - \sum \tan A \tan B}$$

14. Summation of Series

$$\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n-1)\beta) = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}$$

$$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (n-1)\beta) = \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}$$

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