

CHAPTER

13



Three Dimensional Geometry

1. Vector Representation of a Point : Position vector of point

$$P(x, y, z) \text{ is } x\hat{i} + y\hat{j} + z\hat{k}.$$

2. Distance Formula

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, AB = |\overline{OB} - \overline{OA}|$$

3. Distance of P from Coordinate Axes

$$PA = \sqrt{y^2 + z^2}, PB = \sqrt{z^2 + x^2}, PC = \sqrt{x^2 + y^2}$$

4. Section Formula : $x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n}, z = \frac{mz_2 + nz_1}{m+n}$

Mid Point : $x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}, z = \frac{z_1 + z_2}{2}$

5. Direction Cosines and Direction Ratios

(i) **Direction cosines :** Let α, β, γ be angles which a directed line makes with the positive direction of the axes of x, y and z respectively, then $\cos \alpha, \cos \beta, \cos \gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n) . Thus $\ell = \cos \alpha, m = \cos \beta, n = \cos \gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$.

(iii) **Direction ratios:** Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\ell = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

(v) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1, b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of

line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}, m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$.

6. Angle between Two Line Segments

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

The line will be perpendicular if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$,

parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

7. Projection of a Line Segment on a Line : If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then the projection of PQ on a line having direction cosines ℓ, m, n is $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$.

8. Equation of a Plane : General form : $ax + by + cz + d = 0$, where a, b, c are not all zero, $a, b, c, d \in R$.

(i) Normal form : $\ell x + my + nz = p$

(ii) Plane through the point (x_1, y_1, z_1) : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

(iii) Intercept form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(iv) Vector form: $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ or $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

(v) Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.

Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

(vi) **Equation of a plane passing through a given point and parallel to the given vectors:** $\vec{r} = \vec{a} + \lambda \vec{b} + \mu \vec{c}$ (parametric form) where λ & μ are scalars.

or $\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$ (non parametric form)

9. A Plane and a Point

(i) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\frac{ax' + by' + cz' + d}{\sqrt{a^2 + b^2 + c^2}}$.

(ii) Length of the perpendicular from a point (\vec{a}) to plane $\vec{r} \cdot \vec{n} = d$ is given by $p = \frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$.

(iii) Foot (x', y', z') of perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is given by $\frac{x' - x_1}{a} = \frac{y' - y_1}{b} = \frac{z' - z_1}{c} = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$.

(iv) **To find image of a point w.r.t. a plane:** Let $P(x_1, y_1, z_1)$ is a given point and $ax + by + cz + d = 0$ is given plane. Let (x', y', z') is the image point then

$$\frac{x' - x_1}{a} = \frac{y' - y_1}{b} = \frac{z' - z_1}{c} = -2 \frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

10. Angle between Two Planes: $\cos \theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2} \sqrt{a'^2 + b'^2 + c'^2}}$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

The angle θ between the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by, $\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|}$.

Planes are perpendicular if $\vec{n}_1 \cdot \vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda \vec{n}_2$ (λ is a non zero scalar.)

11. Angle Bisectors

(i) The equations of the planes bisecting the angle between two given planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(ii) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \Rightarrow$ origin lies in acute angle

12. Family of Planes

(i) Any plane through the intersection of

$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

(ii) The equation of plane passing through the intersection of the planes $\vec{r} \cdot \vec{n}_1 = d_1$ & $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

13. Area of Triangle : From two vector \vec{AB} and \vec{AC} . Then area is given by $\frac{1}{2} |\vec{AB} \times \vec{AC}|$.

14. Volume of a Tetrahedron: Volume of a tetrahedron with vertices $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ and

$D(x_4, y_4, z_4)$ is given by $V = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

A Line

1. Equation of a Line

(i) A straight line is intersection of two planes. It is represented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

(ii) **Symmetric form:** $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$.

(iii) **Vector equation:** $\vec{r} = \vec{a} + \lambda \vec{b}$.

(iv) Reduction of cartesian form of equation of a line to vector form and vice versa

$$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow \vec{r} = (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}) + \lambda (a \hat{i} + b \hat{j} + c \hat{k})$$

2. Angle between a Plane and a Line

(i) If θ is the angle between line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$, Then

$$\sin \theta = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)} \sqrt{l^2 + m^2 + n^2}}$$

(ii) Vector form: If θ is the angle between a line

$\vec{r} = (\vec{a} + \lambda \vec{b}) = \vec{r} \cdot \vec{n} = d$ then $\sin \theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| |\vec{n}|}$.

(iii) Condition for perpendicularity $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}, \vec{b} \times \vec{n} = 0$.

(iv) Condition for parallel $al + bm + cn = 0, \vec{b} \cdot \vec{n} = 0$.

3. Condition for a Line to Lie in a Plane

(i) **Cartesian form:** $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ would lie in a plane

$ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $\frac{a}{l} + \frac{b}{m} + \frac{c}{n} = 0$.

(ii) **Vector form:** $\vec{r} = \vec{a} + \lambda \vec{b}$ would line in the plane $\vec{r} \cdot \vec{n} = d$ if $\vec{b} \cdot \vec{n} = 0$ & $\vec{a} \cdot \vec{n} = d$.

4. Skew Lines

(i) The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines. If

$$\begin{vmatrix} \alpha' - \alpha & \beta' - \beta & \gamma' - \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} \neq 0. \text{ Then lines are skew.}$$

(ii) **Vector Form:** For lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ to be skew $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) \neq 0$.

(iii) Shortest distance between line $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ &

$\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$.

5. Sphere: General equation of a sphere is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

$(-u, -v, -w)$ is the centre and $\sqrt{u^2 + v^2 + w^2 - d}$ is the radius of the sphere.

6. Volume of tetrahedron = 1/3 × height × Area of base

$= \frac{1}{6}$ [Area of parallelepiped]