

CHAPTER

10

Permutations and Combinations

**Fundamental Principle of Counting
(Counting without actually counting)**

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of different ways of

- Simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).
- Happening of exactly one of the events is $m + n$ (known as addition principle).

Factorial

A Useful Notation : $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$;

$$n! = n \cdot (n-1)! \text{ where } n \in \mathbb{W}$$

$$0! = 1! = 1$$

$$(2n)! = 2^n \cdot n! [1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)]$$

Permutation

- ${}^n P_r$ denotes the number of permutations of n different things, taken r at a time ($n \in \mathbb{N}$, $r \in \mathbb{W}$, $n \geq r$)

$${}^n P_r = n(n-1)(n-2) \dots (n-r+1) = \frac{n!}{(n-r)!}$$

- The number of permutations of n things taken all at a time when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining

$$n - (p + q + r) \text{ are all different is : } \frac{n!}{p!q!r!}.$$

- The number of permutation of n different objects taken r at a time, when a particular object is always to be included is $r! \cdot {}^{n-1} C_{r-1}$.
- The number of permutation of n different object taken r at a time, when repetition be allowed any number of times is $n \times n \times n \dots r \text{ times} = n^r$.
- (i) The number of circular permutations of n different

things taken all at a time is ; $(n-1)! = \frac{{}^n P_n}{n}$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

- The number of circular permutation of n different things taking r at a time distinguishing clockwise & anticlockwise arrangement is $\frac{{}^n P_r}{r}$.

Combination

- ${}^n C_r$ denotes the number of combinations of n different things taken r at a time, and ${}^n C_r = \frac{n!}{r!(n-r)!} = \frac{{}^n P_r}{r!}$ where

$r \leq n$; $n \in \mathbb{N}$ and $r \in \mathbb{W}$. ${}^n C_r$ is also denoted by $\binom{n}{r}$ or A_r^n or $C(n, r)$.

- The number of combination of n different things taking r at a time.

(i) When p particular things are always to be included $= {}^{n-p} C_{r-p}$.

(ii) When p particular things are always to be excluded $= {}^{n-p} C_r$.

(iii) When p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q} C_{r-p}$.

- Given n different objects, the number of ways of selecting atleast one of them is, ${}^n C_1 + {}^n C_2 + {}^n C_3 + \dots + {}^n C_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

- (i) Total number of ways in which it is possible to make a selection by taking some or all out of $p + q + r + \dots$ things, where p are alike of one kind, q a like of a second kind, r alike of third kind and so on is given by : $(p+1)(q+1)(r+1) \dots - 1$.

(ii) The total number of ways of selecting one or more things from p identical of one kind, q identical things of second kind, r identical things of third kind and n different things is $(p+1)(q+1)(r+1) 2^n - 1$.

Divisors

Let $N = p^a \cdot q^b \cdot r^c \dots$ where $p, q, r \dots$ are distinct primes and $a, b, c \dots$ are natural numbers then :

- The total numbers of divisors of N including 1 & N is $= (a+1)(b+1)(c+1) \dots$

- (b) The sum of these divisors is $= (p^0 + p^1 + p^2 + \dots + p^a)(q^0 + q^1 + q^2 + \dots + q^b)(r^0 + r^1 + r^2 + \dots + r^c) \dots$
- (c) Number of ways in which N can be resolved as a product of two factor is =
- $$\frac{1}{2} (a + 1)(b + 1)(c + 1) \dots$$
- if N is not a perfect square
- $$\frac{1}{2} [(a + 1)(b + 1)(c + 1) \dots + 1]$$
- if N is a perfect square
- (d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N .

Division and Distribution

- (a) (i) The number of ways in which $(m + n)$ different things can be divided into two groups containing m and n things respectively is :
- $$\frac{(m + n)!}{m!n!} \quad (m \neq n).$$
- (ii) If $m = n$, it means the groups are equal & in this case the number of subdivision is $\frac{(2n)!}{n!n!2!}$; for in any one way it is possible to inter change the two groups without obtaining a new distribution.
- (iii) If $2n$ things are to be divided equally between two persons then the number of ways is $\frac{(2n)!}{n!n!(2!)}$ $\times 2$
- (b) (i) Number of ways in which $(m + n + p)$ different things can be divided into three groups containing m, n and p things respectively is $\frac{(m + n + p)!}{m!n!p!}$, $m \neq n \neq p$.
- (ii) If $m = n = p$ then the number of groups = $\frac{(3n)!}{n!n!n!3!}$.
- (iii) If $3n$ things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

- (c) In general, the number of ways of dividing n distinct object into l groups containing p objects each, m groups containing q objects each is equal to $\frac{n!(l + m)!}{(p!)^l (q!)^m l! m!}$.
- Here $lp + mq = n$.
- (d) Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them = p^n .
- (e) Number of ways in which n identical things may be distributed among p persons if each person may receive one, one or more things is; ${}^{n+p-1}C_n$.

Dearrangement

Number of ways in which n letters can be placed in n directed envelopes so that no letter goes into its own envelope is

$$= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!} \right]$$

Important Result

- (a) Number of rectangle of any size in a square of size $n \times n$ is $\sum_{r=1}^n r^3$ and number of square of any size is $\sum_{r=1}^n r^2$.
- (b) Number of rectangle of any size in a rectangle of size $n \times p$ ($n < p$) is $\frac{np}{2} (n + 1)(p + 1)$ and number of squares of any size is $\sum_{r=1}^n (n + 1 - r)(p + 1 - r)$.
- (c) If there are n points in a plane of which m ($< n$) are collinear:
- (i) Total number of lines obtained by joining these points is ${}^n C_2 - {}^m C_2 + 1$.
- (ii) Total number of different triangle ${}^n C_3 - {}^m C_3$.
- (d) Maximum number of point of intersection of n circles is ${}^n P_2$ and n lines is ${}^n C_2$.