CHAPTER 15

Parabola

General Equation of A Conic : Focal Directrix Property

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is:

$$(l^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2\,(lx + my + n)^2$$
$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Case (i) When the focus lines on the directrix

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if:

$e > 1$, $h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1$, $h^2 = ab$ the lines will coincident.

$e < 1$, $h^2 < ab$ the lines will be imaginary.

When the focus does not lie on the directix

The conic represents:

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1$; $D \neq 0$ $h^2 = ab$	$0 < e < 1$; $D \neq 0$ $h^2 < ab$	$D \neq 0$; $e > 1$ $h^2 > ab$	$e > 1$; $D \neq 0$ $h^2 > ab$; $a + b = 0$

Standard equation of a parabola is $y^2 = 4ax$. For this parabola:

(*i*) Vertex is (0, 0)

(*ii*) Focus is $(a, 0)$

(*iii*) Axis is $y = 0$

(*iv*) Directrix is $x + a = 0$

Latus Rectum

A focal chord perpendicular to the axis of a parabola is called the LATUS RECTUM. For $y^2 = 4ax$.

(*i*) Length of the latus rectum $= 4a$.

(*ii*) Length of the semi latus rectum $= 2a$.

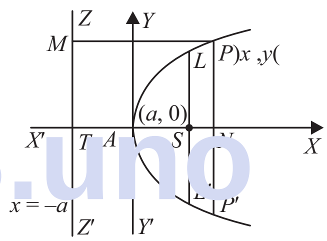
(*iii*) Ends of the latus rectum are $L(a, 2a)$ & $L'(a, -2a)$.

Parametric Representation

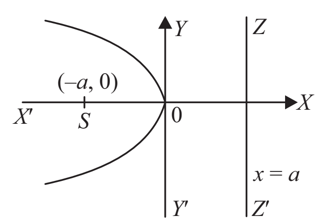
The simplest & the best form of representing the co-ordinates of a point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

Types of Parabola

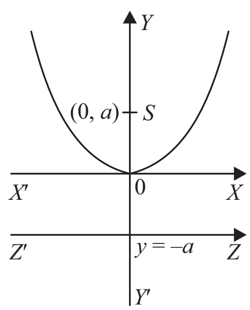
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$.



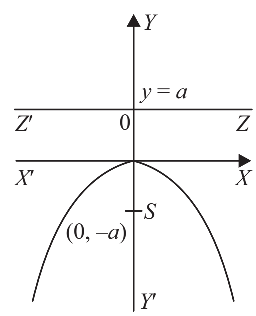
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Para-metric equation	Focal length
$y^2 = 4ax$	(0, 0)	$(a, 0)$	$y = 0$	$x = -a$	$4a$	$(a, \pm 2a)$	$(at^2, 2at)$	$x + a$
$y^2 = -4ax$	(0, 0)	$(-a, 0)$	$y = 0$	$x = a$	$4a$	$(-a, \pm 2a)$	$(-at^2, 2at)$	$x - a$
$x^2 = +4ay$	(0, 0)	$(0, a)$	$x = 0$	$y = -a$	$4a$	$(\pm 2a, a)$	$(2at, at^2)$	$y + a$
$x^2 = -4ay$	(0, 0)	$(0, -a)$	$x = 0$	$y = a$	$4a$	$(\pm 2a, -a)$	$(2at, -at^2)$	$y - a$
$(y-k)^2 = 4a(x-h)$	(h, k)	$(h + a, k)$	$y = k$	$x + a - h = 0$	$4a$	$(h + a, k \pm 2a)$	$(h + at^2, k + 2at)$	$x - h + a$
$(x-p)^2 = 4b(y-q)$	(p, q)	$(p, b + q)$	$x = p$	$y + b - q = 0$	$4b$	$(p \pm 2a, q + a)$	$(p + 2at, q + at^2)$	$y - q + b$

Position of a Point Relative to a Parabola

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

Chord Joining Two Points

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.

Note:

(*i*) If PQ is focal chord then $t_1t_2 = -1$.

(*ii*) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

(*iii*) If $t_1t_2 = k$ then chord always passes a fixed point $(-ka, 0)$.

Line & A Parabola

(*a*) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a > = < cm$

$\Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

Note: Line $y = mx + c$ will be tangent to parabola

$x^2 = 4ay$ if $\mathbf{c = -am^2}$.

(*b*) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1 + m^2)(a - mc)}$.

Note: length of the focal chord making an angle α with the x-axis is $4a \operatorname{cosec}^2 \alpha$.

Tangent to the Parabola $y^2 = 4ax$

(*a*) **Point form:** Equation of tangent to the given parabola at its point (x_1, y_1) is $yy_1 = 2a\,(x + x_1)$.

(*b*) **Slope form:** Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(*c*) **Parametric form:** Equation of tangent to the given parabola at its point $P(t)$, is–

$$ty = x + at^2$$

Note: Point of intersection of the tangents at the point t_1 & t_2 is $[at_1\,t_2, a(t_1 + t_2)]$. (i.e. G.M. and A.M. of abscissae and ordinates of the points).

Normal to the Parabola $y^2 = 4ax$

(*a*) **Point form:** Equation of normal to the given parabola at its point (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

(*b*) **Slope form:** Equation of normal to the given parabola whose slope is 'm', is $y = mx - 2am - am^3$ foot of the normal is $(am^2, -2am)$.

(*c*) **Parametric form:** Equation of normal to the given parabola at its point $P(t)$, is $y + tx = 2at + at^3$.

Note:

(*i*) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$.

(*ii*) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(*iii*) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

Chord of Contact

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$. Also note that the chord of contact exists only if the point P is not inside.

Chord with A Given MIddle Point

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$.

Diameter

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

Conormal Points

Foot of the normals of three concurrent normals are called conormals point.

(*i*) Algebraic sum of the slopes of three concurrent normals of parabola $y^2 = 4ax$ is zero.

(*ii*) Sum of ordinates of the three conormal points on the parabola $y^2 = 4ax$ is zero.

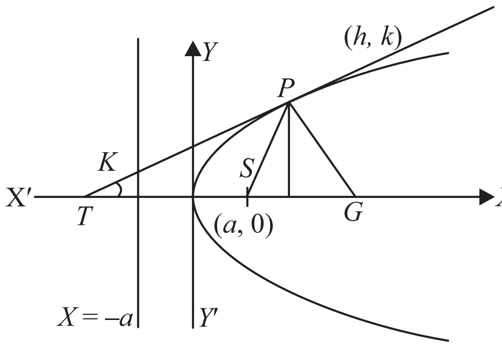
(*iii*) Centroid of the triangle formed by three co-normal points lies on the axis of parabola.

(*iv*) If $27ak^2 < 4(h - 2a)^3$ satisfied then three real and distinct normal are drawn from point (h, k) on parabola $y^2 = 4ax$.

(*v*) If three normals are drawn from point $(h, 0)$ on parabola $y^2 = 4ax$, then $h > 2a$ and one the normal is axis of the parabola and other two are equally inclined to the axis of the parabola.

Important Highlights

(*a*) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then $ST = SG = SP$ where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.



(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix**, and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of a length $a\sqrt{1+t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean beetween segments of any focal chord

i.e. $2a = \dfrac{2bc}{b+c}$ or $\dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a}$.

(f) Image of the focus lies on directrix with respect to any tangent of parabola $y^2 = 4ax$.