

CHAPTER

5

Limits of Functions

Limit

Limit of a function $f(x)$ is said to exist as $x \rightarrow a$ when,

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = M \text{ some finite value } M.$$

(Left hand limit) (Right hand limit)

Indeterminate Forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad (\infty) - (\infty)$$

$$\infty \times 0, \quad (1)^\infty, \quad (0)^0, \quad (\infty)^0$$

Standard Limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x}$$

$$= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1,$$

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e, \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a, \quad a > 0,$$

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

Note

$$\log_a x \lll a^x \lll x! \quad \begin{matrix} a > 1 \\ x \in \mathbb{N} \end{matrix}$$

Fundamental Theorems on Limits

Let $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} g(x) = m$. If l and m exists finitely then:

- (a) Sum rule: $\lim_{x \rightarrow a} [f(x) + g(x)] = l + m$
- (b) Difference rule: $\lim_{x \rightarrow a} [f(x) - g(x)] = l - m$
- (c) Product rule: $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = l \cdot m$
- (d) Quotient rule: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{l}{m}$, provided $m \neq 0$

(e) Power rule: If m and n are integers, then

$$\lim_{x \rightarrow a} [f(x)]^{m/n} = l^{m/n}, \text{ provided } l^{m/n} \text{ is a real number.}$$

(f) $\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(m)$; provided $f(x)$ is continuous at $x = m$.

Limits Using Expansion

(i) $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \dots, a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, for $-1 < x \leq 1$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, for $-1 < x \leq 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vii) $\sin^{-1} x = x + \frac{1^2}{3!} x^3 + \frac{1^2 \cdot 3^2}{5!} x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!} x^7 + \dots$

(viii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

(xi) For $|x| < 1, n \in \mathbb{R}, (1+x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \dots$

(xii) $(1+x)^{1/x} = e^{\frac{1}{x} \ln(1+x)} = e \left[1 - \frac{x}{2} + \frac{11}{24} x^2 - \frac{21}{48} x^3 + \dots + \infty \right]$

Limits of form 1^∞ , 0^0 , ∞^0 .

Also for $(1)^\infty$ type of problems we can use following rules.

$$(a) \lim_{x \rightarrow 0} (1+x)^{1/x} = e,$$

$$(b) \lim_{x \rightarrow a} [f(x)]^{g(x)}, \text{ where } f(x) \rightarrow 1; g(x) \rightarrow \infty \text{ as } x \rightarrow a \text{ then}$$

$$\lim_{x \rightarrow a} [f(x)]^{g(x)} = e^{\lim_{x \rightarrow a} \{f(x)-1\}g(x)}$$

Sandwich Theorem or Squeeze Play Theorem

If $f(x) \leq g(x) \leq h(x) \forall x$ and $\lim_{x \rightarrow a} f(x) = l = \lim_{x \rightarrow a} h(x)$, then $\lim_{x \rightarrow a} g(x) = l$

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