

CHAPTER

17



Hyperbola

❖ Standard equation of the hyperbola is  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ,  
 where  $b^2 = a^2 (e^2 - 1)$   
 or  $a^2 e^2 = a^2 + b^2$  i.e.  $e^2 = 1 + \frac{b^2}{a^2} = 1 + \left( \frac{\text{Conjugate Axis}}{\text{Transverse Axis}} \right)^2$

(a) Foci:

$S \equiv (ae, 0)$  &  $S' \equiv (-ae, 0)$ .

(b) Equations of Directrices:

$x = \frac{a}{e}$  &  $x = -\frac{a}{e}$ .

(c) Vertices:

$A \equiv (a, 0)$  &  $A' \equiv (-a, 0)$ .

(d) Latus Rectum:

(i) Equation:  $x = \pm ae$

(ii) Length =  $\frac{2b^2}{a} = \frac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a (e^2 - 1)$   
 $= 2e(\text{distance from focus to directrix})$

(iii) Ends:  $\left( ae, \frac{b^2}{a} \right), \left( ae, -\frac{b^2}{a} \right); \left( -ae, \frac{b^2}{a} \right), \left( -ae, -\frac{b^2}{a} \right)$

(e) Focal Property:

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e.  $||PS| - |PS'|| = 2a$ . The distance  $SS' =$  focal length.

(f) Focal Distance:

Distance of any point  $P(x, y)$  on hyperbola from foci  $PS = ex - a$  &  $PS' = ex + a$ .

**Conjugate Hyperbola:**  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  &  $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  are conjugate hyperbolas of each.

**Auxillary Circle:**  $x^2 + y^2 = a^2$ .

**Parametric Representation:**  $x = a \sec \theta$  &  $y = b \tan \theta$

**Position of A point 'P' w.r.t. A Hyperbola:**

$S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > =$  or  $< 0$  according as the point  $(x_1, y_1)$  lies inside, on or outside the curve.

Tangents

(i) Slope Form:  $y = m \times \pm \sqrt{a^2 m^2 - b^2}$

(ii) Point Form: at the point  $(x_1, y_1)$  is  $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ .

(iii) Parametric Form:  $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$ .

❖ Normal to The Hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ :

(a) Point form: Equation of the normal to the given hyperbola at the point  $P(x_1, y_1)$  on it is  $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2 = a^2 e^2$ .

(b) Slope form: The equation of normal of slope  $m$  to the given hyperbola is  $y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{(a^2 - m^2 b^2)}}$  foot of normal are

$$\left( \pm \frac{a^2}{\sqrt{(a^2 - m^2 b^2)}}, \mp \frac{mb^2}{\sqrt{(a^2 - m^2 b^2)}} \right)$$

(c) Parametric form: The equation of the normal at the point  $P (a \sec \theta, b \tan \theta)$  to the given hyperbola is

$$\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2 = a^2 e^2.$$

Director Circle

Equation to the director circle is:  $x^2 + y^2 = a^2 - b^2$ .

Chord of Contact

If  $PA$  and  $PB$  be the tangents from point  $P(x_1, y_1)$  to the Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , then the equation of the chord of contact  $AB$  is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \text{ or } T = 0 \text{ at } (x_1, y_1).$$

Equation of Chord with mid Point  $(x_1, y_1)$

The equation of the chord of the ellipse  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , whose mid-

point be  $(x_1, y_1)$  is  $T = S_1$  where  $T$

$$= \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1, S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$



$$\text{i.e. } \left( \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 \right) = \left( \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 \right).$$

### Asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

**Reflection property of the hyperbola:** An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

**Rectangular or Equilateral Hyperbola:**  $xy = c^2$ , eccentricity is  $\sqrt{2}$ .

**Vertices:**  $(\pm c, \pm c)$ ; **Foci:**  $(\pm \sqrt{2}c, \pm \sqrt{2}c)$ . **Directrices:**  $x + y = \pm \sqrt{2}c$ .

**Latus Rectum (l):**  $l = 2\sqrt{2}c = T.A. = C.A.$

Parametric equation  $x = ct, y = c/t, t \in R - \{0\}$

Equation of the tangent at  $P(x_1, y_1)$  is  $\frac{x}{x_1} + \frac{y}{y_1} = 2$  & at  $P(t)$  is  $\frac{x}{t} + ty = 2c$ .

Equation of the normal at  $P(t)$  is  $xt^3 - yt = c(t^4 - 1)$ .

Chord with a given middle point as  $(h, k)$  is  $kx + hy = 2hk$ .

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