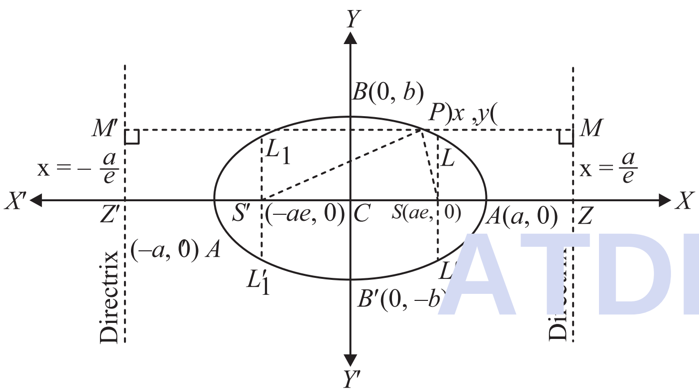
CHAPTER 16

Ellipse

The co-ordinate axis is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Where $a > b$ and $b^2 = a^2(1 - e^2)$

$\Rightarrow \quad a^2 - b^2 = a^2 e^2$.

where e = eccentricity ($0 < e < 1$).



FOCI : S = (ae, 0) and S′ ≡ (– ae, 0).

(*j*) **Latus Rectum:** The focal chord perpendicular to the major axis is called the **latus rectum**.

(*i*) Length of latus rectum

$$(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1 - e^2)$$

(*ii*) Equation of latus rectum : $x = \pm ae$.

(*iii*) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right)$,

$L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(*k*) **Focal Radii:** SP = $a - ex$ and $S'P$ = a + ex

$\Rightarrow \quad SP + S'P = 2a$ = Major axis.

(*l*) **Eccentricity:** e = $\sqrt{1 \quad \frac{b^2}{a^2}}$

Position of a Point W.r.t. an Ellipse

The point $P(x_1, y_1)$ lines outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > <$ or $= 0$.

Paramatric Representation

The equations $x = a \cos \theta$ and $y = b \sin \theta$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where θ is a parameter (eccentric angle).

Note that if P(θ) ≡ (a cos θ, b sin θ) is on the ellipse then ; Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

Line and an Ellipse

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two real points, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α and β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

Tangent to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(*a*) **Point form:** Equation of tangent to the given ellipse at its point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

(*b*) **Slope form:** Equation of tangent to the given ellipse whose slope is 'm', $y = mx \pm \sqrt{a^2m^2 + b^2}$.

Point of contact are $\left(\frac{\pm a^2 m}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}}\right)$

(*c*) **Parametric form:** Equation of tangent to the given ellipse at its point (a cos θ, b sin θ), is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

Normal to the Ellipse $\frac{x}{a^2}+\frac{y}{b^2}=1$

(a) Point form: Equation of the normal to the given ellipse at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.

(b) Slope form: Equation of a normal to the given ellipse whose slope is 'm' is $y=mx\pm\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.

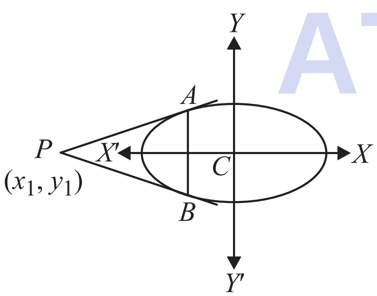
(c) Parametric form: Equation of the normal to the given ellipse at the point ($a\cos\theta$, $b\sin\theta$) is ax. sec θ – by. cosec θ = (a^2-b^2).

Chord of Contact

If PA and PB be the tangents from point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or $T=0$ at (x_1, y_1).

Pair or Tangents

If $P(x_1, y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and a pair of tangents PA, PB can be drawn to it from P.



Then the equation of pair of tangents of PA and PB is $SS_1=T^2$

where $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

i.e., $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$

Director Circle

$x^2+y^2=a^2+b^2$ i.e. a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

Euqation of Chord with Mid Point (x_1, y_1)

i.e. $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

Important Highlights

1. If P be any point on the ellipse with S and S' as its foci then $\ell(SP)+\ell(S'P)=2a$.
2. The locus of the point of interseciton of feet of perpendicular from foci on any tangent to an ellipse is the auxiliary circle.
3. The product of perpendicular distance from the foci to any tangent of an ellipse is equal to square of the semi minor axis.
4. Tangents at the extremities of latus-rectum of an ellipse intersect on the foot of corresponding directrix.
5. The portion of the tangent to an ellipse between the point of contact and the directrix subtends a right angle at the corresponding focus.
6. Tangent and normal at any point P bisect the external and internal angles between the focal distances of SP and $S'P$.
7. If the normal at any point P on the ellipse with centre C meet the major and minor axes in G and g respectively and if CF be perpendicular upon this normal then

   (i) $PF\cdot PG=b^2$ (ii) $PF\cdot Pg=a^2$
8. Area enclosed by an ellipse having length of major and minor axes as $2a$ and $2b$ is given by πab.