

## CHAPTER

## 4

## Determinants

## Definition

1. The determinant consisting two rows and two columns is

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \text{ its value is given by:}$$

$$D = a_1b_2 - a_2b_1$$

2. A determinant which consists of three rows and three columns is called a third-order-determinant.

$$\text{Let } D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then its value is}$$

$$D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}$$

## Minors and Cofactors

$$\text{If } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then the minor } M_{ij} \text{ of the element } a_{ij} \text{ is the}$$

determinant obtained by deleting the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column,

$$\text{i.e. } M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

The **cofactor** of the element  $a_{ij}$  is  $C_{ij} = (-1)^{i+j} M_{ij}$ .

## Properties of Minors and Cofactors

1. The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero,

$$\text{i.e., if } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0 \text{ and so on.}$$

2. The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) is  $\Delta$ ,

$$\text{i.e., if } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$$|A| = \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}.$$

3. In general, if  $|A| = \Delta$ , then  $|A| = \sum_{i=1}^n a_{ij} C_{ij}$  and  $|(\text{adj } A)| = \Delta^{n-1}$ , where  $A$  is a matrix of order  $n \times n$ .

## Properties of Determinants

- The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows.  
e.g.,  $|A'| = |A|$
- If  $A = [a_{ij}]_{n \times n}$ ,  $n > 1$  and  $B$  be the matrix obtained from  $A$  by interchanging two of its rows or columns, then  $\det(B) = -\det(A)$
- If two rows (or columns) of a square matrix  $A$  are proportional, then  $|A| = 0$ .
- $|kA| = k^n |A|$  where  $B$  is the matrix obtained from  $A$ , by multiplying one row (or column) of  $A$  by  $k$ .
- $|kA| = k^n |A|$ , where  $A$  is a matrix of order  $n \times n$ .
- If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinant, e.g.,

$$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

- If the same multiple of the elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged, e.g.,

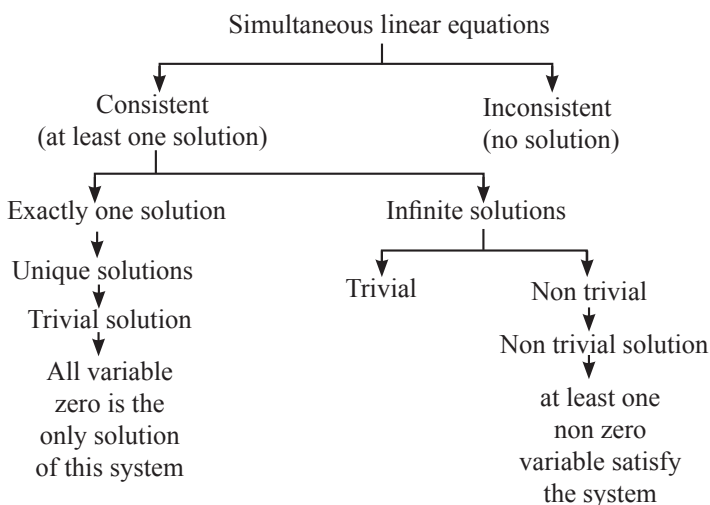
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + ka_{31} & a_{12} + ka_{32} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- If each element of a row (or column) of a determinant is zero, then its value is zero.
- If any two rows (or columns) of a determinant are identical, then its value is zero.
- If  $r$  rows (or  $r$  columns) become identical, when  $a$  is substituted for  $x$ , then  $(x - a)^{r-1}$  is a factor of given determinant.

## Important Results on Determinants

- $|AB| = |A| |B|$ , where  $A$  and  $B$  are square matrices of the same order.
- $|A^n| = |A|^n$ .
- If  $A, B$  and  $C$  are square matrices of the same order such that  $i^{\text{th}}$  columns (or rows) of  $A$  is the sum of  $i^{\text{th}}$  columns (or rows) of  $B$  and  $C$  and all other columns (or rows) of  $A, B$  and  $C$  are identical, then  $|A| = |B| + |C|$ .
- $|I_n| = 1$ , where  $I_n$  is identity matrix of order  $n$ .
- $|O_n| = 0$ , where  $O_n$  is a zero matrix of order  $n$ .
- If  $\Delta(x)$  has a third order determinant having polynomials as its elements.
  - If  $\Delta(a)$  has two rows (or columns) proportional, then  $(x - a)$  is a factor of  $\Delta(x)$ .
  - If  $\Delta(a)$  has three rows (or columns) proportional, then  $(x - a)^2$  is a factor of  $\Delta(x)$ .
- A square matrix  $A$  is non-singular, if  $|A| \neq 0$  and singular, if  $|A| = 0$ .
- Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.
- In general,  $|B + C| \neq |B| + |C|$ .
- Determinant of a diagonal matrix = Product of its diagonal elements.
- If  $A$  is a non-singular matrix, then  $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$ .
- Determinant of an orthogonal matrix =  $\pm 1$ .
- Determinant of a hermitian matrix is purely real.
- If  $A$  and  $B$  are non-zero matrices and  $AB = O$ , then it implies  $|A| = 0$  or  $|B| = 0$ .

## System of Equation



## Cramer's Rule: [Simultaneous Equations Involving Three Unknowns]

$$\begin{aligned} \text{Let } a_1x + b_1y + c_1z &= d_1 && \dots(i) \\ a_2x + b_2y + c_2z &= d_2 && \dots(ii) \\ a_3x + b_3y + c_3z &= d_3 && \dots(iii) \end{aligned}$$

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}; D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\& D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

### Note:

- If  $D \neq 0$  and atleast one of  $D_1, D_2, D_3 \neq 0$ , then the given system of equations are consistent and have unique non trivial solution.
- If  $D \neq 0$  &  $D_1 = D_2 = D_3 = 0$ , then the given system of equations are consistent and have trivial solution only
- If  $D = D_1 = D_2 = D_3 = 0$ , then the given system of equations are consistent and have infinite solution.

## Applications of Determinants in Geometry

Let the three points in a plane be  $A(x_1, y_1), B(x_2, y_2)$  and  $C(x_3, y_3)$ , then

$$1. \text{ Area of } \Delta ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$2. \text{ If the given points are collinear, then } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

3. Let two points are  $A(x_1, y_1), B(x_2, y_2)$  and  $P(x, y)$  be a point on the line joining points  $A$  and  $B$ , then the equation of line is

$$\text{given by } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

