

CHAPTER

9



Definite Integration

The Fundamental Theorem of Calculus Part 1:

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t) dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) = f(x)$.

The Fundamental Theorem of Calculus, Part 2:

If f is continuous on $[a, b]$, then $\int_a^b f(x) dx = F(b) - F(a)$

where F is any antiderivative of f , that is, a function such that $F' = f$.

Note: If $\int_a^b f(x) dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has at least one root lying in (a, b) provided f is a continuous function in (a, b) .

✦ $\int_a^b f(x) dx =$ algebraic area under the curve $f(x)$ from a to b

Properties of Definite Integral

$$1. \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$2. \int_a^b f(x) dx = -\int_b^a f(x) dx$$

$$3. \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$4. \int_{-a}^a f(x) dx = \int_0^a (f(x) + f(-x)) dx = \begin{cases} 2 \int_0^a f(x) dx, & f(-x) = f(x) \\ 0, & f(-x) = -f(x) \end{cases}$$

$$5. \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$6. \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$7. \int_0^{2a} f(x) dx = \int_0^a (f(x) + f(2a-x)) dx$$

$$= \begin{cases} 2 \int_0^a f(x) dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

8. If $f(x)$ is a periodic function with period T , then

$$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z},$$

$$\int_a^{a+nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z}, a \in \mathbb{R}$$

$$\int_0^{nT} f(x) dx = (n-m) \int_0^T f(x) dx, m, n \in \mathbb{Z},$$

$$\int_{nT}^{(n+m)T} f(x) dx = \int_0^m f(x) dx, n \in \mathbb{Z}, a \in \mathbb{R}$$

$$\int_{a+nT}^{b+nT} f(x) dx = \int_a^b f(x) dx, n \in \mathbb{Z}, a, b \in \mathbb{R}$$

9. If $\psi(x) \leq f(x) \leq \phi(x)$ for $a \leq x \leq b$, then

$$\int_a^b \psi(x) dx \leq \int_a^b f(x) dx \leq \int_a^b \phi(x) dx$$

Leibnitz Theorem

$$\text{If } F(x) = \int_{g(x)}^{h(x)} f(t) dt,$$

$$\text{then } \frac{dF(x)}{dx} = h'(x)f(h(x)) - g'(x)f(g(x))$$

Walli's Formula

$$1. \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{(n-1)(n-3)\dots(1 \text{ or } 2)}{n(n-2)\dots(1 \text{ or } 2)} K$$

$$\text{where } K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$$2. \int_0^{\pi/2} \sin^n x \cdot \cos^m x \, dx$$

$$= \frac{[(n-1)(n-3)(n-5)\dots 1 \text{ or } 2][(m-1)(m-3)\dots 1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)\dots 1 \text{ or } 2} K$$

$$\text{where } K = \begin{cases} \frac{\pi}{2} & \text{if both } m \text{ and } n \text{ are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$$

Definite Integral as Limit of a Sum

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a + \overline{n-1}h)]$$

$$\Rightarrow \lim_{h \rightarrow \infty} h \sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x) \, dx \text{ where } b-a = nh$$

$$\text{If } a=0 \text{ and } b=1 \text{ then, } \lim_{n \rightarrow \infty} h \sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x) \, dx; \text{ where } nh=1$$

$$\text{OR } \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x) \, dx.$$

Estimation of Definite Integral

1. If $f(x)$ is continuous in $[a, b]$ and it's range in this interval is $[m, M]$, then $m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a)$
2. If $f(x) \leq \phi(x)$ for $a \leq x \leq b$ then $\int_a^b f(x) \, dx \leq \int_a^b \phi(x) \, dx$
3. $\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$
4. If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x) \, dx \geq 0.$
5. $f(x)$ and $g(x)$ are two continuous function on $[a, b]$ then $\left| \int_a^b f(x)g(x) \, dx \right| \leq \sqrt{\int_a^b f^2(x) \, dx} \sqrt{\int_a^b g^2(x) \, dx}$

Some Standard Results

1. $\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2 = \int_0^{\pi/2} \log \cos x \, dx$
2. $\int_a^b \frac{|x|}{x} \, dx = |b| - |a|.$

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