QUADRATIC EQUATION

* Algebraic expression

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x^1 + a_0 x^0$$

(i) $a_n \neq 0$ (ii) Power of x is whole number

is called polynomial.

eg: $x^3 - 7x^2 + \pi x + 5$, $x^7 - 6x^5 + ex^{-3} + 7$, $ex^3 - \pi x^2 + \sin x$

✓ ✗ ✗

* $a_n \to$ leading coefficient * $a_n x^n \to$ leading term.

If leading coefficient is ~~called~~ '1' the polynomial is called as monic polynomial.

eg: $5x^6 - 7x + 8 \to$ Monic polynomial ✗

$5x^2 - 6x + 3x^7 + x^8 \to$ Monic polynomial ✓

$\to$ Highest power of x in polynomial

Degree	Name	General Form	Example
(undefined)	Zero polynomial	$0 \cdot x^{2,3,5,\dots}$	0
0	(Non-zero) constant polynomial	$a; (a \neq 0)$	$-2, -1, 1, 7, 3$
1	Linear polynomial	$ax + b; (a \neq 0)$	$x + 1$
2	Quadratic polynomial	$ax^2 + bx + c; (a \neq 0)$	$x^2 + 1$
3	Cubic polynomial	$ax^3 + bx^2 + cx + d; (a \neq 0)$	$x^3 + 1$
4	Bi-Quadratic	$ax^4 + bx^3 + cx^2 + dx + e; (a \neq 0)$	$x^4 + x^3 + 1$

$$ax^2+bx+c=0,\ (a\neq 0)$$

$$\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$\left(\frac{\text{coeff. of } x}{2}\right)^2$ ↳ add + Subtract

$$\Rightarrow x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$b^2-4ac=D=\text{Discriminant}$$

$$\Rightarrow x=\frac{-b\pm\sqrt{D}}{2a}\quad \alpha=\frac{-b+\sqrt{D}}{2a},\ \beta=\frac{-b-\sqrt{D}}{2a}$$

algebra

⇕

No: of roots of a poly: of degree n is exactly n in number real or imaginary counted with multiplicity

$(x-1)^3=0$

↳ cubic

→ solution = 1 (one)

→ roots = 1, 1, 1 (three)

* Relation b/w roots

(1) Sum of roots

$$\alpha+\beta=\frac{-b+\sqrt{D}}{2a}+\frac{-b-\sqrt{D}}{2a}=\frac{-2b}{2a}=\frac{-b}{a}$$

$$\Rightarrow S=\alpha+\beta=\frac{-b}{a}$$

(2) Products of roots

$$P=\alpha\cdot\beta=\left(\frac{-b+\sqrt{D}}{2a}\right)\left(\frac{-b-\sqrt{D}}{2a}\right)$$

$$=\frac{b^2-D}{4a^2}=\frac{\cancel{b^2}-\cancel{b^2}+4ac}{4a^2}$$

$$\Rightarrow P=\alpha\cdot\beta=\frac{c}{a}$$

β → Roots

$$P(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a\left(x^2 - (\alpha+\beta)x + \alpha\beta\right)$$
$$= a\left(x^2 - \alpha x - \beta x + \alpha\beta\right)$$
$$= a\left(x(x-\alpha) - \beta(x-\alpha)\right)$$
$$= a(x-\alpha)(x-\beta)$$

* $x^2 + x + 1$
$$= (x-\omega)(x-\omega^2)$$

* $x^2 + 1 = 0$
$$x^2 = -1$$
$$x = \pm i$$
$$\therefore x^2 + 1 = (x+i)(x-i)$$

* Nature of roots

* if $D > 0 \Rightarrow$ roots are real and distinct

$$\left\{ x = \frac{-b \pm \sqrt{D}}{2a} \quad \alpha = \frac{-b+\sqrt{D}}{2a}, \quad \beta = \frac{-b-\sqrt{D}}{2a} \right\}$$

* if $D = 0 \Rightarrow$ roots are real and equal

$$\left\{ \therefore \alpha = \beta = \frac{-b}{2a} \right\}$$

* if $D < 0 \Rightarrow$ imaginary roots

Note: for real roots $D \geq 0$

$$(\alpha-\beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$
$$= (\alpha+\beta)^2 - 4\alpha\beta$$
$$= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a}$$
$$= \frac{b^2}{a^2} - \frac{4c}{a}$$
$$(\alpha-\beta)^2 = \frac{b^2 - 4ac}{a^2}$$
$$(\alpha-\beta)^2 = \frac{D}{a^2} \quad [\text{Always Applicable}]$$

* Some important points

P(1): if $a, b, c \in \mathbb{Q}$ and D is perfect square then roots are rational.

Ex: $x^2 - 2\sqrt{3}x - 1 = 0$ [if $a, b, c \notin \mathbb{Q}$]

$$x = \frac{2\sqrt{3} \pm \sqrt{16}}{2} = \sqrt{3} \pm 2$$

↳ Not Rational

$$|\alpha - \beta| = \frac{\sqrt{D}}{|a|} \quad [\text{Not valid}]$$

$x^2 + 1 = 0 \rightarrow D = -4$, $a = 1$

$\alpha = i \quad \beta = -i$

$$|i - (-i)|$$
$$= |2i| = 2 \neq \frac{\sqrt{-4}}{1}$$

~~D~~ then roots are integers.

even + even = even
odd − even = odd

Proof: $x^2+bx+c=0$, $b, c \in I$, D is square of an integer

$$x = \frac{-b \pm \sqrt{D}}{2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

b = even integer ⇒ $x = \frac{\overset{even}{-b} \pm \overset{even}{\sqrt{b^2-4c}}}{\underset{\hookrightarrow\, even}{2}}$ = integer

b = odd integer: $\frac{\overset{odd}{-b} \pm \boxed{\sqrt{b^2-4c}} \rightarrow odd}{2}$ = integer

$b^2 - 4c$: even − ... = odd; 2 ↓ even

odd − odd = even

P(3): $a, b, c \in Q$, D is not perfect square but $D>0$ then roots are irrational and occurs in conjugate pair of surds. i.e form of $p+\sqrt{q}$, $p-\sqrt{q}$

$$x = \frac{-b+\sqrt{D}}{2a}, \frac{-b-\sqrt{D}}{2a}$$

$$= \frac{-b}{2a} + \frac{\sqrt{D}}{2a}, \frac{-b}{2a} - \frac{\sqrt{D}}{2a}$$

surds
↳ √ of perfect Square नहीं ~~होते~~ होते.

conjugate
↳ जिस Number से multiply करने पर Rational no. आयेगी

P(4): $a, b, c \in R$, $D<0$
then ~~roo~~ imaginary roots and occurs in conjugate pairs. i.e one roots is $p+iq$ then other root is $p-iq$

Note: $a, b, c \in R$ → one root is $a+ib$ then 2nd root is $a-ib$

if $a, b, c \in Q$ → one root of quadratic is $3+\sqrt{7}$ then other root is $3-\sqrt{7}$

$x^2+x+1=0$

$$x = \frac{-1 \pm \sqrt{3}i}{2}$$

$$x = \underbrace{\frac{-1+\sqrt{3}i}{2}}_{\omega}, \underbrace{\frac{-1-\sqrt{3}i}{2}}_{\omega^2}$$

are $1, \frac{c}{a}$.

Proof $a(1)^2+b(1)+c=0 \Rightarrow$ is root:

$\therefore ax^2+bx+c=0 \begin{cases} 1 \\ \beta \end{cases}$

$\therefore P = 1\cdot\beta = \frac{c}{a} \Rightarrow \left(\beta = \frac{c}{a}\right)$

P(6): if $a-b+c=0$ then roots of quad. $ax^2+bx+c=0$ are $-1, -\frac{c}{a}$

Proof $\therefore ax^2+bx+c=0 \begin{cases} -1 \\ \beta \end{cases}$

$\therefore P = (-1)(\beta) = \frac{c}{a} \Rightarrow \left(\beta = -c/a\right)$

* Equation from roots:

roots $= \alpha, \beta$

$\therefore$ eq^n = $(x-\alpha)(x-\beta)=0$

$\Rightarrow x^2 - \beta x - \alpha x + \alpha\beta = 0$

$\Rightarrow x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - Sx + P = 0$

$\begin{bmatrix} S = \alpha+\beta \\ P = \alpha\cdot\beta \end{bmatrix}$

Q: If $p(q-r)x^2+q(r-p)x+r(p-q)$ has equal roots prove $\frac{2}{q} = \frac{1}{p}+\frac{1}{r}$

M-1 $D=0$ {roots equal}

$D = (q(r-p))^2 - 4p(q-r)r(p-q) = 0$

$\Rightarrow (\underbrace{-q(p-r)}_{A+B})^2 - 4\underbrace{p(q-r)}_{A}\cdot\underbrace{r(p-q)}_{B} = 0$

$A = p(q-r) = pq - pr$

$B = r(p-q) = rp - qr$

$A+B = pq - qr$

$A+B = q(p-r)$

$\Rightarrow (A+B)^2 - 4AB = 0$

$\Rightarrow A^2+B^2+2AB-4AB=0$

$\Rightarrow (A-B)^2 = 0 \Rightarrow \{A=B\}$

$\therefore p(q-r) = r(p-q)$

$pq - pr = pr - qr$

$2pr = pq + qr$

$\overline{pqr}\left(\frac{2}{q} = \frac{1}{r} + \frac{1}{p}\right.$ Q.E.D

$$\underbrace{\quad}_{a} \qquad \underbrace{\quad}_{b} \qquad \underbrace{\quad}_{c}$$

$$\Rightarrow a+b+c = \cancel{pq} - pr + qr - \cancel{qp} + rp - rq = 0$$

$\Downarrow$

roots are $1, \frac{c}{a}$

we given roots are equal

$$1 = \frac{c}{a} \Rightarrow c = a \Rightarrow r(p-q) = p(q-r)$$

$$pr - qr = pq - pr$$

$$2pr = pq + qr$$

$$\overline{pqr} \quad \left\{ \frac{2}{q} = \frac{1}{r} + \frac{1}{p} \right\}$$

Q: If α, β are roots of $x^2-2x+5=0$ then form a quadratic equation whose roots are $\alpha^3+\alpha^2-\alpha+22$ & $\beta^3+4\beta^2-4\beta+35$

$x^2-2x+5=0 \rightarrow \alpha, \beta$ — to form a quadratic whose roots are $\alpha^3+\alpha^2-\alpha+22$, $\beta^3+4\beta^2-4\beta+35$

$$\left. \begin{array}{l} \alpha^2-2\alpha+5=0 \\ \beta^2+2\beta+5=0 \end{array} \right]$$

जब Higher degree से Lower degree में जाते हैं तो Division method से !!

सोच

$$p(x) = \underbrace{q(x)}_{0}\cdot g(x) + r(x)$$
$$= r(x)$$

$$\begin{array}{r|l} \alpha^2-2\alpha+5 & \alpha^3+\alpha^2-\alpha+22 \quad (\alpha+3 \\ & \underline{\alpha^3-2\alpha^2+5\alpha} \\ & 3\alpha^2-6\alpha+22 \\ & \underline{3\alpha^2-6\alpha+15} \\ & 7 \end{array}$$

$\therefore \alpha^3+\alpha^2-\alpha+22 = (\alpha^2-2\alpha+5)(\alpha+3) + 7$

$\alpha^3+\alpha^2-\alpha+22 = +7$ (one root)

$$\begin{array}{r|l} \beta^2+2\beta+5 & \beta^3+4\beta^2-7\beta+35 \quad (\beta \\ & \underline{\beta^3+2\beta^2+5\beta} \\ & 6\beta^2-12\beta+35 \\ & \underline{6\beta^2-12\beta+30} \\ & 5 \end{array}$$

$\therefore \beta^3-4\beta^2-7\beta+35 = (\beta^2+2\beta+5)(\beta+6) + 5$

$\rightarrow \beta^3-4\beta^2-7\beta+35 = +5$ (2nd root)

$\therefore$ Eqⁿ with roots 5 & 7

$$\therefore x^2-12x+35=0$$

If $f(\alpha,\beta) = f(\beta,\alpha)$ then $f(\alpha,\beta)$ is symmetric funcⁿ of roots

eg: $f(\alpha,\beta) = \alpha^2+\beta^2,\ \alpha^2\beta+\beta^2\alpha + \frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$f(\beta,\alpha) = \beta^2+\alpha^2 + \beta^2\alpha+\alpha^2\beta + \frac{\beta}{\alpha}+\frac{\alpha}{\beta}$

Note: Every symmetric function in α,β can be expressed in terms of two symmtric function.

$\alpha+\beta$ and $\alpha\cdot\beta$

NICHOD! Value of symmetric Expression in α,β can be found using sum of roots & Product of roots.

* A Golden Point:

(i) if $D_1+D_2 \geq 0 \Rightarrow$
→ Atleast one equation has real roots.
→ if roots of one equation are imaginary then those of other equation will be real & unequal

(ii) if $D_1+D_2<0 \Rightarrow$
→ Atleast one of the equation has imaginary roots.
→ if roots of one equation are real then the other equation will be Imaginary.

* Quadratic Equation v/s Identity:

If a quadratic has more than 2 distinct roots then it becomes an identity

eg) $(a+b)^2 = a^2+b^2+2ab$ (Identity) $\sec^2\theta - \tan^2\theta = 1\ \theta \neq (2n+1)\frac{\pi}{2}$ (Identity)

$2a+3=5$ (Eqⁿ)

Proof: $ax^2+bx+c=0$ → α, β, γ → distinct

$a\alpha^2+b\alpha+c=0$
$a\beta^2+b\beta+c=0$
$a\gamma^2+b\gamma+c=0$

→ $a(\alpha^2-\beta^2)+b(\alpha-\beta)=0 \Rightarrow a(\alpha+\beta)+b=0$

→ $a(\beta^2-\gamma^2)+b(\beta-\gamma)=0 \Rightarrow a(\beta+\gamma)+b=0$

$a(\alpha-\gamma)=0 \Rightarrow a=0$
$b=0$
$c=0$

Now. Quad. become

$0\cdot x^2+0\cdot x+0=0$ (Identity).

* In any polynomial equation, if the ... then it becomes an identity.

* In any polynomial equation becomes an identity, then its all coefficients are simultaneously zero

↳ eg:) $\{0 \cdot x^3 - 0 \cdot x^2 + 0 \cdot x + 0 = 0\}$

* Newton's Formula:→

$ax^2+bx+c=0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \beta\end{smallmatrix}$, $S_n = \alpha^n + \beta^n$, $T_n = \alpha^n - \beta^n$

$a\alpha^2 + b\alpha + c = 0 \times \alpha^{n-2} \Rightarrow a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2} = 0$

$a\beta^2 + b\beta + c = 0 \times \beta^{n-2} \Rightarrow a\beta^n + b\beta^{n-1} + c\beta^{n-2} = 0$

Add $a(\alpha^n+\beta^n) + b(\alpha^{n-1}+\beta^{n-1}) + c(\alpha^{n-2}+\beta^{n-2}) = 0$

$$\Rightarrow \boxed{aS_n + bS_{n-1} + c\cdot S_{n-2} = 0}$$

subtract: $a(\alpha^n-\beta^n) + b(\alpha^{n-1}-\beta^{n-1}) + c(\alpha^{n-2}-\beta^{n-2}) = 0$

$$\boxed{aT_n + bT_{n-1} + cT_{n-2} = 0}$$

Q $ax^2+bx+c=0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \beta\end{smallmatrix}$ find $\alpha^4+\beta^4 = ??$

$aS_n + bS_{n-1} + cS_{n-2} = 0$

$n=4$: $aS_4 + bS_3 + cS_2 = 0$ —(1)

$n=3$: $aS_3 + bS_2 + cS_1 = 0$

$S_1 = \alpha+\beta$

$S_2 = \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

⇓

S_3 → from 1st eqⁿ We get S_4.

Q $x^2-6x-2=0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \beta\end{smallmatrix}$, $a_n = \alpha^n - \beta^n$ then find $\dfrac{a_{10}-2a_8}{2a_9}$

$x^2-6x-2=0$

$\Rightarrow a_n - 6a_{n-1} - 2a_{n-2} = 0$

$\boxed{n=10}$ $a_{10} - 6a_9 - 2a_8 = 0$

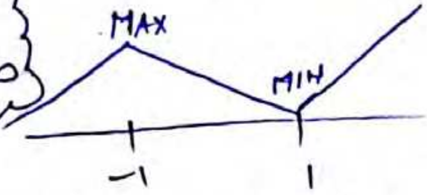
$\Rightarrow a_{10} - 2a_8 = 3\cdot 2a_9 \Rightarrow \dfrac{a_{10}-2a_8}{2a_9} = 3$ Ans.

Graph: $y = x + \frac{1}{x}$

$\hookrightarrow \frac{dy}{dx} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$

$\frac{1}{\text{very small}} \to \infty$ = very small +ve no.

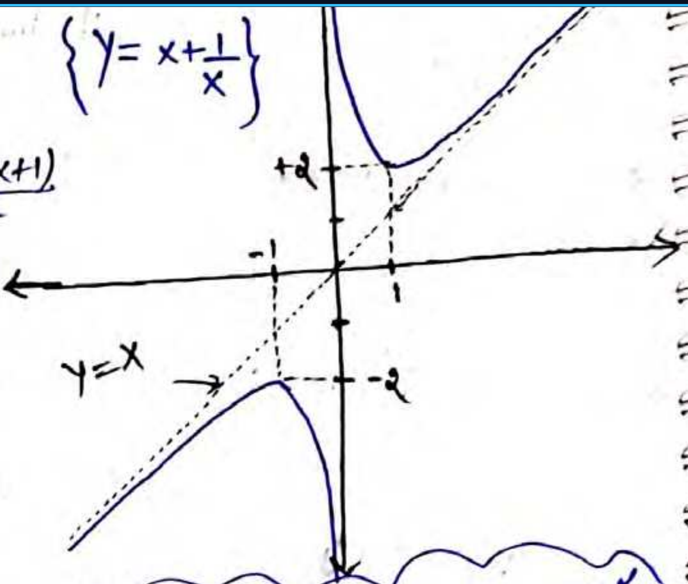
$\frac{1}{\text{very small}} \to -\infty$ = very small -ve no.



$\lim_{x \to 0^+} x + \frac{1}{x} = 0 + \infty = \infty$

$\lim_{x \to 0^-} x + \frac{1}{x} = -\infty$

$\{ y = x + \frac{1}{x} \}$



$a(x-\alpha)(x-\beta) = ax^2 + bx + c \to \alpha_1, \alpha_2$

$\alpha_1 + \alpha_2 = -b/a$, $\alpha_1 \alpha_2 = \frac{c}{a}$

* General Polynomial Equation:-

$a_0x^3 + a_1x^2 + a_2x + a_3 = 0 \to \alpha_1, \alpha_2, \alpha_3$

$a_0x^3 + a_1x^2 + a_2x + a_3 = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$

$\cancel{= a_0(x^3 - x^2(\alpha_1+\alpha_2+\alpha_3) + x(\alpha_1\alpha_2+\alpha_2\alpha_3}$

$= a_0(x^3 - x^2(\alpha_1+\alpha_2+\alpha_3) + x(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) - \alpha_1\alpha_2\alpha_3)$

$= a_0x^3 - a_0x^2(\alpha_1+\alpha_2+\alpha_3) + a_0x(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1) - a_0\alpha_1\alpha_2\alpha_3$

coefficient of x^2: $a_1 = -a_0(\alpha_1+\alpha_2+\alpha_3)$

coefficient of x: $a_2 = a_0(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)$

constant term: $a_3 = -a_0\alpha_1\alpha_2\alpha_3$

$\therefore S_1 = \alpha_1 + \alpha_2 + \alpha_3 = (-1)^1 \frac{a_1}{a_0}$

$S_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = (-1)^2 \frac{a_2}{a_0}$

$S_3 = \alpha_1\alpha_2\alpha_3 = (-1)^3 \frac{a_3}{a_0}$

S_1, S_3, S_5 ---- sab minus ke saath aatey hai.

S_2, S_4, S_6 ---- yeh positive ke saath.

eg: $2x^3 - 7x^2 + 6x + 5 \to \alpha, \beta, \gamma$

$S_1 = \alpha + \beta + \gamma = -(-7/2) = 7/2$

$S_2 = \alpha\beta + \beta\gamma + \gamma\alpha = 6/2 = 3$

$S_3 = \alpha_1\alpha_2\alpha_3 = -5/2$

$\alpha_2, \alpha_3, \alpha_4$

$$S_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4 = -\left(\frac{a_1}{a_0}\right)$$

$$S_2 = \alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4 = \left(\frac{a_2}{a_0}\right)$$

$$S_3 = \alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_3\alpha_4 = -\left(\frac{a_3}{a_0}\right)$$

$$S_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{a_4}{a_0}$$

$$\therefore a_0x^n+a_1x^{n-1}+a_2x^{n-2}+a_3x^{n-3}+\dots+a_{n-1}x+a_n=0 \to \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$$

$$S_1 = \alpha_1+\alpha_2+\alpha_3+\dots+\alpha_n = -\frac{a_1}{a_0}$$

$$S_2 = \text{product taken two at a time} = \frac{a_2}{a_0}$$

$\vdots$

$$S_n = \alpha_1\alpha_2\alpha_3\dots\alpha_n = (-1)^n\left(\frac{a_n}{a_0}\right)$$

NOTE: Every odd degree polynomial equation with real coefficient must have at least one real root. Because imaginary roots occur in conjugate pair.

eg. $ax^3+bx^2+cx+d=0 \to 3+2i, 5, 7$ ✗ $\quad ax^3+bx^2+cx+d=0 \to 3+2i, 3-2i, 5$ ✓

$(a,b,c,d \in R)$

eg Possible Cases for 5 degree eq^n
- All 5 roots real ✓
- 4 real & 1 imaginary ✗
- 3 real & 2 imaginary ✓
- 2 real & 3 imaginary ✗
- 1 real & 4 imaginary ✓
- All 5 imaginary. ✗

* Transformation of Equation:

$x^2-5x+6=0 \to 2=\alpha,\ 3=\beta$ form a quad. whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$

$\frac{1}{\alpha}, \frac{1}{\beta}$; $f(\alpha)=\frac{1}{\alpha}$, $f(\beta)=\frac{1}{\beta}$

Put $y=f(x)=\frac{1}{x} \Rightarrow x=\frac{1}{y}$ put in original eq^n.

$\therefore f(x)=\frac{1}{x}$ (माना)

$$\therefore \frac{1}{y^2}-\frac{5}{y}+6=0$$

$$\Rightarrow 6y^2-5y+1=0 \xrightarrow[y \leftrightarrow x]{\text{Replace}} 6x^2-5x+1$$

$$a_1x + b_1y + c_1 = 0 \times b_2$$
$$a_2x + b_2y + c_2 = 0 \times b_1$$
$$\overline{(a_1b_2 - b_1a_2)x + b_2c_1 - b_1c_2 = 0}$$

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - b_1a_2} = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$a_1x + b_1y + c_1 = 0 \times a_2$$
$$a_2x + b_2y + c_2 = 0 \times a_1$$
$$\overline{(b_1a_2 - a_1b_2)y + a_2c_1 - a_1c_2 = 0}$$

$$y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - b_1a_2} = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Finally if $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

Then
$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

* Condition for common roots

$$\left.\begin{aligned} a_1x^2 + b_1x + c_1 = 0 \\ a_2x^2 + b_2x + c_2 = 0 \end{aligned}\right\} \rightarrow \text{Have a common root } \alpha$$

$$\therefore \left.\begin{aligned} a_1\alpha^2 + b_1\alpha + c_1 = 0 \\ a_2\alpha^2 + b_2\alpha + c_2 = 0 \end{aligned}\right\} \quad \alpha^2 = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad \alpha = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\therefore \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^{\cancel{2}}} = \frac{\begin{vmatrix} b & c_1 \\ b_2 & c_2 \end{vmatrix}}{\cancel{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}} \Rightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2$$

Finally

if $\begin{aligned} a_1x^2 + b_1x + c_1 = 0 \\ a_2x^2 + b_2x + c_2 = 0 \end{aligned}$ have a common root α

then
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2$$

→ This is condition for atleast one root common root.

$\therefore\ a_1x^2+b_1x+c_1=0 <^{\alpha}_{\beta} \rightarrow S=\alpha+\beta=\frac{-b_1}{a_1}\quad P=\frac{c_1}{a_1}$

$a_2x^2+b_2x+c_2=0 <^{\alpha}_{\beta} \rightarrow S=\alpha+\beta=\frac{-b_2}{a_2}\quad P=\alpha\cdot\beta=\frac{c_2}{a_2}$

Hence condition for Both roots common.

$$\Downarrow$$

$$+\frac{b_1}{a_1}=+\frac{b_2}{a_2}\ \Big\}\ \frac{c_1}{a_1}=\frac{c_2}{a_2}$$

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\quad\rightarrow\quad\frac{a_1}{a_2}=\frac{b_1}{b_2}\ \Big\}\ \frac{a_1}{a_2}=\frac{c_1}{c_2}$$

NOTE:

$a_1x^2+b_1x+c_1=0 <^{p+iq}_{p-iq}$

$a_2x^2+b_2x+c_2=0 <^{p+iq}_{p-iq}$

Have a common roots & the roots of one of the equations are imaginary then both roots will be common

if the equation

$F(x)=0\rightarrow\alpha$

$g(x)=0\rightarrow\alpha$

Have a common root say α then $x=\alpha$ is also $Af(x)\pm Bg(x)=0\rightarrow\alpha$

eg: $x^2-5x+6=0 <^{2}_{3}$

$x^2-6x+8=0 <^{4}_{2}$

$\underline{-\quad+\quad-}$

$x-2=0 \Rightarrow \boxed{x=2}$

Q ~~let α, β~~, $x^3+2x^2-4x+5=0 \rightarrow \alpha, \beta, \gamma$. find $\frac{(\alpha^3+5)(\beta^3+5)(\gamma^3+5)}{13\alpha\cdot\beta\cdot\gamma}$

$\alpha^3+5=4\alpha-2\alpha^2=2\alpha(2-\alpha)$

$\alpha^3+5=2\alpha(2-\alpha)$

lly $\beta^3+5=2\beta(2-\beta)$

$\gamma^3+5=2\gamma(2-\gamma)$

$$E=\frac{(\alpha^3+5)(\beta^3+5)(\gamma^3+5)}{13\alpha\cdot\beta\cdot\gamma}$$

$$=\frac{2\cancel{\alpha}(2-\alpha)\cdot2\cancel{\beta}(2-\beta)\cdot2\cancel{\gamma}(2-\gamma)}{13\cancel{\alpha}\cdot\cancel{\beta}\cdot\cancel{\gamma}}$$

put eqn (1)

$$\Rightarrow\frac{8(2-\alpha)(2-\beta)(2-\gamma)}{13}$$

$$\Rightarrow\frac{8\times13}{13}=\boxed{8}$$

MAST TARIKA

$x^3+2x^2-4x+5=1\cdot(x-\alpha)(x-\beta)(x-\gamma)$

$\alpha\quad\beta\quad\gamma$

put $x=2$

$8+8-8+5=(2-\alpha)(2-\beta)(2-\gamma)$

$\Rightarrow(2-\alpha)(2-\beta)(2-\gamma)=13$ —(1)

$f(x) = ax^2 + bx + c = a(x-\alpha)(x-\beta)$

if $D = 0 \Rightarrow \boxed{\alpha = \beta}$

$\therefore f(x) = a(x-\alpha)(x-\alpha)$

$f(x) = a(x-\alpha)^2$

$a = +ve$

$f(x) = (\sqrt{a})^2(x-\alpha)^2$

$= [\sqrt{a}(x-\alpha)]^2$

$\Downarrow$

f(x) becomes square of a linear polynomial with real coeff.

$a = -ve$

$f(x) = (\sqrt{|a|}\, i)^2 (x-\alpha)^2$

$= [(\sqrt{|a|}\, i)(x-\alpha)]^2$

$(\sqrt{|-4|}\, i)$

$-4 = (2i)^2$

$-16 = (4i)^2$

$\Downarrow$

$(\sqrt{|-16|}\, i)^2$

$\Downarrow$

$(4i)^2$

Q $f(x) = \lambda x^2 - 2\lambda x + 1$

Find λ if (a) f(x) is square of a linear/perfect square

Sol^n $D = 0$

$4\lambda^2 - 4\lambda = 0$ & $a > 0$ $\therefore \lambda > 0$

$4\lambda(\lambda - 1) = 0$

$\lambda = 0, \lambda = 1$

$f(x) = \lambda x^2 - \lambda x - 4$

Find λ if f(x) is square of linear with real coeff.

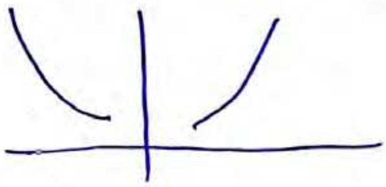
Sol^n $D = 0$

$\lambda^2 + 16\lambda = \lambda(\lambda + 16) = 0$

$\lambda = 0, -16$

ANS: No sol^n value of λ exist!

* Analysis of a Quadratic Polynomial:



$y = f(x) = ax^2 + bx + c, \ (a \neq 0)$

$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$

$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$

$f(x) = y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$

$f\left(\frac{-b}{2a} + k\right) = a\left(\frac{-b}{2a} + k + \frac{b}{2a}\right)^2 - \frac{D}{4a} = ak^2 - \frac{D}{4a}$

$f\left(\frac{-b}{2a} - k\right) = a\left(\frac{-b}{2a} - k + \frac{b}{2a}\right)^2 - \frac{D}{4a} = ak^2 - \frac{D}{4a}$

Same.

$$y = f(x) = ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a}$$

if $a>0$

$y_{max} \to \infty$

$y_{min} \to \frac{-D}{4a}$ at $x = -\frac{b}{2a}$

if $a<0$

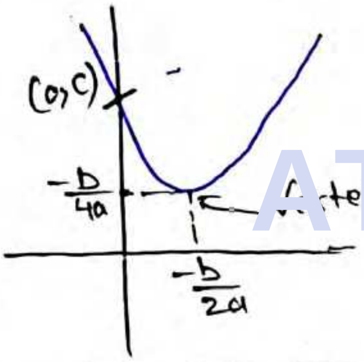
$y_{min} \to -\infty$

$y_{max} = \frac{-D}{4a}$ at $\frac{-b}{2a}$.

$a>0$

$y = ax^2+bx+c$

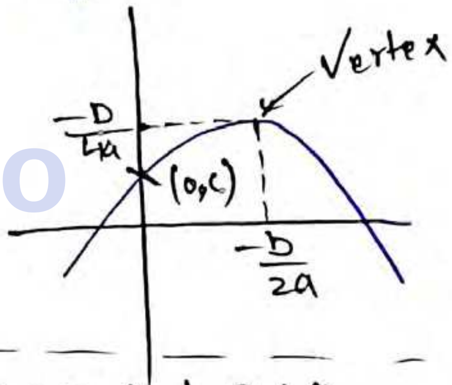
$y_{max} \to \infty$ $\qquad$ $y_{min} = \frac{-D}{4a}$ at $\frac{-b}{2a}$.



$a<0$

$y = ax^2+bx+c$

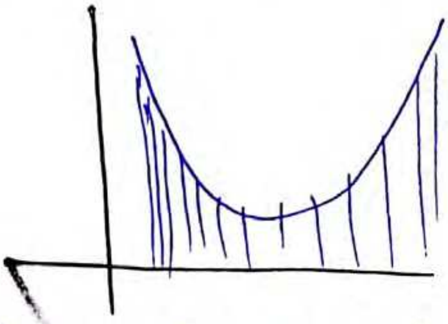
$y_{max} = \frac{-D}{4a}$ at $x = \frac{-b}{2a}$ $\qquad$ $y_{min} \to -\infty$



If $a>0$ but $D<0$ (+ve)

$y_{min} = \frac{-D}{4a}$

$y_{max} \to \infty$



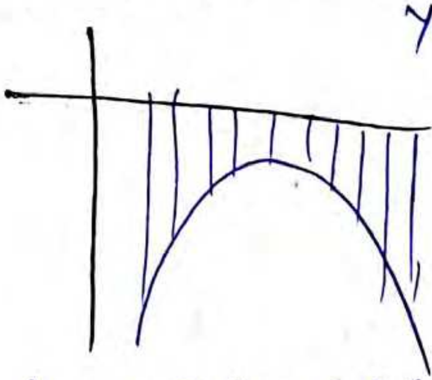
then $ax^2+bx+c>0$

$\forall\ x \in R$

if $a<0$ but $D<0$ (-ve)

$y_{max} = \frac{-D}{4a}$

$y_{min} = \infty$



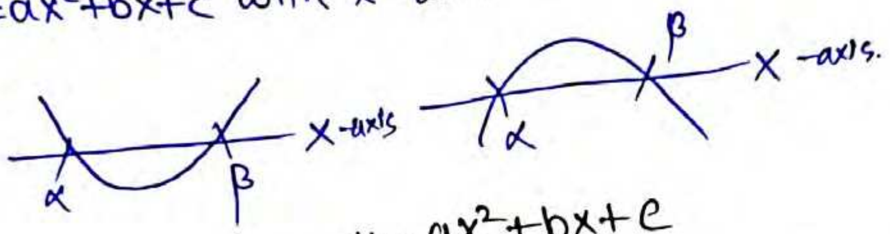
then $ax^2+bx+c<0$

$\forall\ x \in R$

Quadratic Equation:

$y = ax^2 + bx + c \quad (a \neq 0)$

if we put $y = 0$ we get $ax^2 + bx + c = 0$ called Quadratic equation and roots are the nothing but x-coordinates of the point of Intersection of $y = ax^2 + bx + c$ with x-axis.



$y = ax^2 + bx + c$

If $D = 0$

$a > 0$

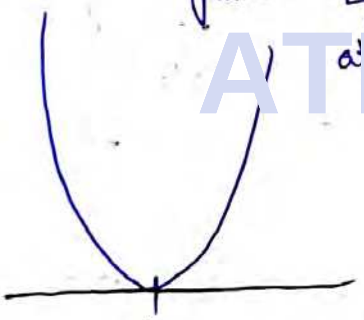
$y_{MAX} \to \infty$

$y_{min} = \frac{-D}{4a} = 0$ at $x = \frac{-b}{2a}$

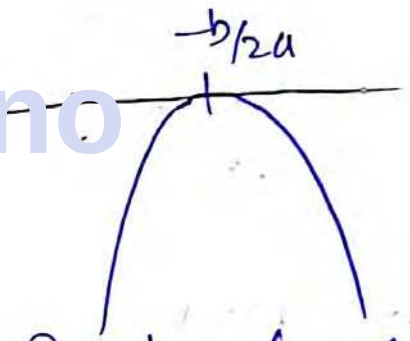
$a < 0$

$y_{MAX} = \frac{-D}{4a} = 0$ at $\frac{-b}{2a}$

$y_{MIN} \to -\infty$



$-b/2a$



Range of $y = (-\infty, 0]$

Range of $y = [0, \infty)$

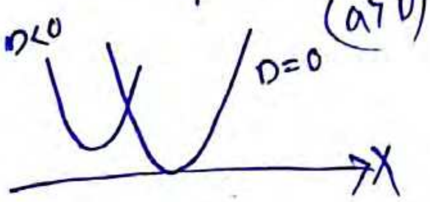
* Important:

2) if $a > 0$ and $D < 0$ then $y = ax^2 + bx + c > 0 \ \forall \ x \in R$

2) if $a < 0$ and $D < 0$ then $y = ax^2 + bx + c < 0 \ \forall \ x \in R$

2) $ax^2 + bx + c \geq 0 \ \forall \ x \in R$ then $a > 0$ & $D \leq 0$

2) $ax^2 + bx + c \leq 0 \ \forall \ x \in R$ then $a < 0$ & $D \leq 0$



Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, then find the value of $P(4)$.
$P(0)=6$, $P(1)=7$, $P(2)=8$ and $P(3)=9$, then find the value of $P(4)$.

Pattern को 4 ...

let $g(x) = P(x) - (x+6) \rightarrow$ polynomial of degree 4

Pattern को पहचान कर Equation form करो!!

$$\left.\begin{aligned} g(0) &= P(0)-6=0 \\ g(1) &= P(1)-7=0 \\ g(2) &= P(2)-8=0 \\ g(3) &= P(3)-9=0 \end{aligned}\right\} \Rightarrow g(0)=g(1)=g(2)=g(3)$$

$$\Downarrow$$

$$g(x) = (x-0)(x-1)(x-2)(x-3)$$

$$P(x) - (x+6) = x(x-1)(x-2)(x-3)$$

$$\Rightarrow P(x) = x(x-1)(x-2)(x-3) + (x+6)$$

$$\Rightarrow P(4) = 4\cdot3\cdot2\cdot1 + 10 = 34$$

* Range of Rational functions :-

Type(1): $f(x) = \dfrac{linear_1}{linear_2}$ i.e $f(x) = \dfrac{ax+b}{cx+d}$ $\left(\dfrac{a}{c} \neq \dfrac{b}{d}\right)$

if $\dfrac{a}{c} = \dfrac{b}{d}$

$f(x) = \dfrac{3x+2}{6x+4}$ $\left(\dfrac{3}{6} = \dfrac{2}{4}\right)$

$= \dfrac{\cancel{3x+2}}{2(\cancel{3x+2})} = 1/2$

$$y = \frac{ax+b}{cx+d} \Rightarrow ycx + yd = ax+b$$

$$\Rightarrow x(cy-a) = b - yd$$

$$\Rightarrow x = \frac{b-yd}{cy-a} \Rightarrow cy-a \neq 0 \Rightarrow y \neq a/c$$

Hence $\Rightarrow$ $y \in R - \left\{\dfrac{a}{c}\right\}$

eg. $f(x) = \dfrac{2x-3}{x-1}$ $\Rightarrow R_f = R - \{2\}$

Type(2): $f(x) = \dfrac{quad_1}{quad_2}$ where numerator & denominator contain a common factor

Steps:

(i) Factorize numerator & Denominator

(ii) Say $(ax-b)$ is a common factor, cancel $(ax-b)$ from numerator & denominator & write $x = b/a$

(iii) Now $f(x) = \dfrac{px+q}{rx+s}$, $x \neq b/a$ & range $= R - \left\{\dfrac{p}{r}, f\left(\dfrac{b}{a}\right)\right\}$

eg: $f(x) = \dfrac{x^2-5x+6}{x^2-6x+8} \Rightarrow f(x) = \dfrac{(x-3)\cancel{(x-2)}}{\cancel{(x-2)}(x-4)}$ $(x \neq 2)$

$f(x) = \dfrac{x-3}{x-4}$, $(x \neq 2)$

if $x = 2$, $f(x) = \dfrac{2-3}{2-4} = 1/2$

Range: $R - \{1, 1/2\}$

* Type (3): $f(x) = \frac{\text{Quad}}{\text{Quad}}, \frac{\text{linear}}{\text{Quad}}, \frac{\text{Quad}}{\text{linear}}$, where numerator & denominator contain no common factor.

Steps:

(i) Equate given expression to y

(ii) Cross multiply & make quad in x

(iii) Since $x \in R$, put $D \geq 0$ & get range of y

(iv) Equate coeff of $x^2 = 0$ to get $y = y_1$ put expression $= y_1$ & solve for x

If we get real x answer of step 3 is final

if we do not get real x exclude y_1 from answer.

Q Find values of 'a' for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

given: Range R, a = ?

$y \in R$

$$y = \frac{ax^2+3x-4}{3x-4x^2+a}$$

$$\Rightarrow y = \frac{ax^2+3x-4}{-4x^2+3x+a}$$

For range to be R: Numerator & denominator should not have a common root

for common root $\begin{vmatrix} a & 3 \\ -4 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & -4 \\ 3 & a \end{vmatrix} = \begin{vmatrix} -4 & a \\ a & -4 \end{vmatrix}^2$

$$\Rightarrow (3a+12)^2 = (16-a^2)^2$$

$$\Rightarrow (3a+12)^2 - (16-a^2)^2 = 0$$

$$\Rightarrow (3a+12+16-a^2)(3a+12-16+a^2) = 0$$

$$\Rightarrow (-a^2+3a+28)(a^2+3a-4) = 0$$

$$\Rightarrow (a^2-3a-28)(a^2+3a-4) = 0$$

$$\Rightarrow a^2-3a-28 = 0, \quad a^2+3a-4 = 0$$

$a = 7, -4$ $\quad$ $a = -4, 1$

$$a \neq -4, 7, 1$$

$3x - 4x^2 + a$

$\Rightarrow 3xy - 4yx^2 + ay = ax^2 + 3x - 4$

$\Rightarrow (a+4y)x^2 + (3-3y)x - 4 - ay = 0$

Since $x \in R$

$D \geq 0$, $9(1-y)^2 + 4(a+4y)(4+ay) \geq 0$

$\Rightarrow 9y^2 + 9 - 18y + 4(4a + a^2y + 16y + 4ay^2) \geq 0$

$\Rightarrow y^2(16a+9) + y(4a^2+46) + 9 + 16a \geq 0 \ \forall\ y \in R$

$D \leq 0$, $16a + 9 > 0$ $\quad (4a^2+46)^2 - 4\cdot(16a+9)(16a+9) \leq 0$, $a > -\frac{9}{16}$ — (i)

$\Rightarrow (4a^2+46)^2 - (2(16a+9))^2 \leq 0$

$\Rightarrow (4a^2+46-32a-18)(4a^2+46+32a+18) \leq 0$

$\Rightarrow (4a^2-32a+28)(4a^2+32a+64) \leq 0$

$\Rightarrow (a^2-8a+7)(a^2+8a+16) \leq 0$

$\Rightarrow (a-1)(a-7)(a+4)^2 \leq 0$

$\Rightarrow (a-1)(a-7) \leq 0$, $a = -4$ is also possible.

$\Rightarrow a \in [1,7] \cup \{-4\}$ — (ii)

From eqn (i) and (ii) $a \in [1,7]$ But $a \neq 7, 1$

$\therefore$ ANS: $a \in (1,7)$

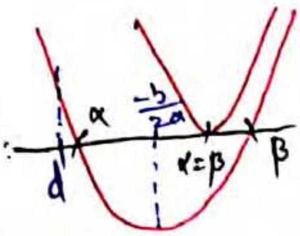
* Location of roots :→

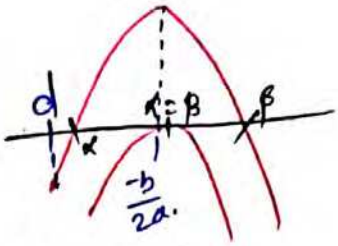
The article deals with an elegant approach of solving problems on quadratic equations when the roots are located/specified on the number line with variety of constraints:

consider: $f(x) = ax^2 + bx + c$

Type 2 : Both roots greater than 'd'

$f(x) = ax^2 + bx + c$





(1) $D \geq 0$
(2) $\frac{-b}{2a} > d$
(3) $f(d) > 0$
(4) $a > 0$

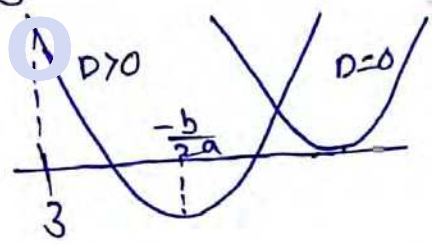
(1) $D \geq 0$
(2) $\frac{-b}{2a} > d$
(3) $f(d) < 0$
(4) $a < 0$

combine

(a) $D \geq 0$ (b) $\frac{-b}{2a} > d$ (c) $a f(d) > 0$

eg: Find all the values of the parameter 'd' for which both roots of the eqn $x^2 - 6dx + (2-2d-9d^2) = 0$ exceed the number 3

$f(x) = x^2 - 6dx + (2-2d-9d^2) = 0$



(1) $D = 36d^2 - 4\cdot(2-2d+9d^2) \geq 0$

$\cancel{9d^2} - 2 + 2d - \cancel{9d^2} \geq 0$

$2d - 2 \geq 0$

$\boxed{d \geq 1}$ — I

(1) $D \geq 0$
(2) $f(3) > 0$
(3) $\frac{-b}{2a} > 3$

(2) $f(3) > 0$

$9 - 18d + 2 - 2d + 9d^2 > 0$

$9d^2 - 20d + 11 > 0$

$9d^2 - 11d - 9d + 11 > 0$

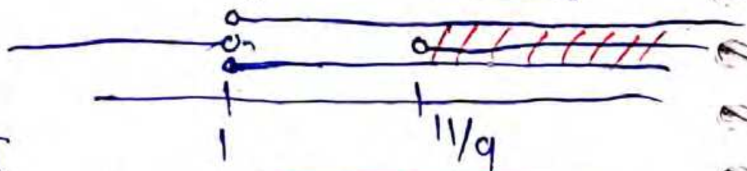
$(9d-11)(d-1) > 0$

$\boxed{d \in (-\infty, 1) \cup (11/9, \infty)}$ — II

(3) $\frac{-b}{2a} > 3$

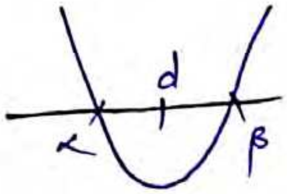
$\Rightarrow \frac{6d}{2} > 3 \Rightarrow 3d > 3$

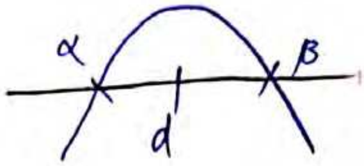
$\Rightarrow \boxed{d > 1}$ — III

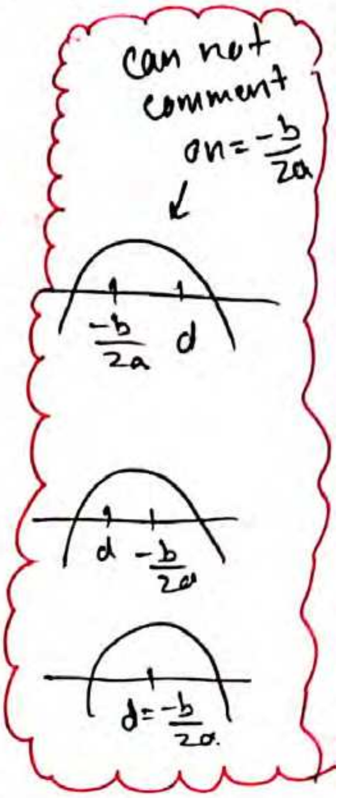


ANS : $\boxed{d \in (11/9, \infty)}$

Type 2: Both roots ... b/w roots or one root is less than 'd' & other root is greater than 'd'







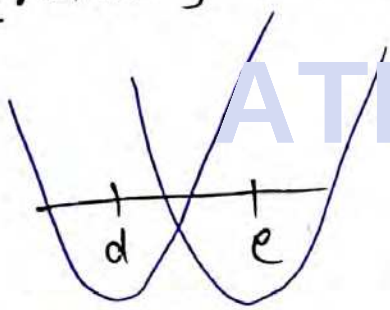
(i) $a > 0$
(ii) $D > 0$
(iii) $f(d) < 0$

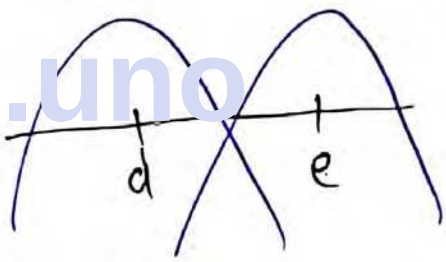
(i) $a < 0$
(ii) $f(d) > 0$
(iii) $D > 0$

combine

(i) $a \cdot f(d) < 0$
(ii) $D > 0$ (No Need)

Type 3: Exactly one root lies b/w d & e $(d < e)$





(i) $f(d) \cdot f(e) < 0$
(ii) $a > 0$
(iii) $D > 0$

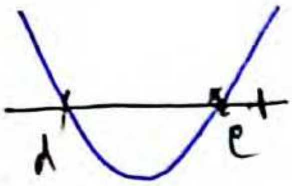
(i) $f(d) \cdot f(e) < 0$
(ii) $a < 0$
(iii) $D > 0$

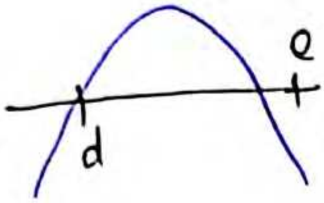
combine

(i) $f(d) \cdot f(e) < 0$
(ii) $a \neq 0$
(iii) $D > 0$ (No Need)

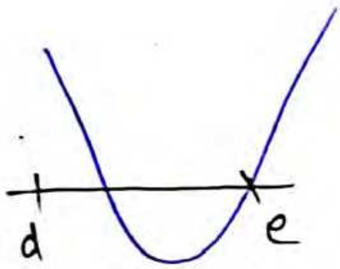
Next

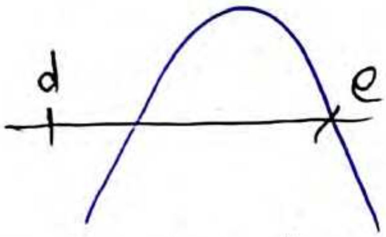
Now ~~....~~ possibilities arise :





One root lies at $x = d$ i.e $f(d) = 0$, then find other root it must lie in (d, e)



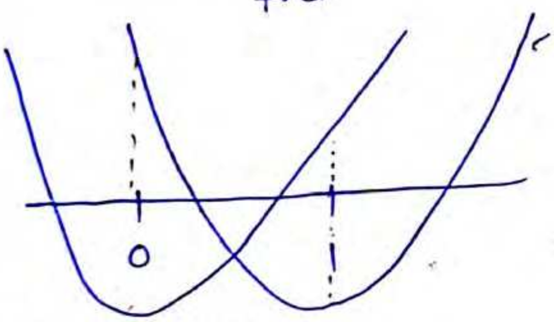


One root lies at $x = e$ i.e $f(e) = 0$ then find the other root it must lie in (d, e)

<u>Question</u> : $(\lambda^2+1)x^2 - 4\lambda x + 2 = 0$ always have exactly one root in the interval $(0,1)$ is: Find set of 'λ'

exactly one root lies in $(0,1)$

$$y = \underbrace{(\lambda^2+1)}_{+ve}x^2 - 4\lambda x + 2 = 0$$



$f(0) \cdot f(1) < 0$

$\Rightarrow 2 \cdot (\lambda^2 + 1 - 4\lambda + 2) < 0$

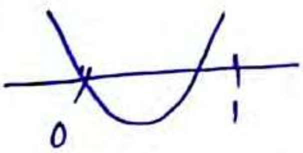
$\Rightarrow (\lambda^2 - 4\lambda + 3) < 0$

$\Rightarrow (\lambda - 3)(\lambda - 1) < 0$

$\Rightarrow \lambda \in (1, 3)$ —— (i)

<u>Now possibility arise</u>

P_1

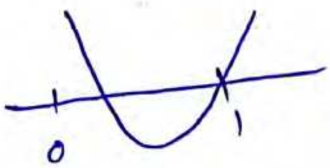


$f(0) = 0 \Rightarrow 2 = 0$

↓

Not possible

P_2



$f(1) = 0 \Rightarrow \lambda^2 + 1 - 4\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\underline{\lambda = 1, 3}$

$2x^2-4x+2=0$

$x^2-2x+1=0$

$x=1,1$

$\therefore \lambda=1$ is rejected

$10x^2-12x+2=0 \to \alpha$

M-1

$5x^2-6x+1=0$

$(5x-1)(x-1)=0$

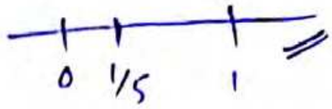
$x=1/5, 1$

M-2 (easier method)

$\alpha\cdot 1 = 2/10$

$x = 1/5$

Hence $\lambda=3$ is accepted



$$\boxed{\therefore\ \lambda\in(1,3)\cup\{3\} \Rightarrow \lambda\in(1,3]}$$

2. IIT can modify the question as!!

Not $(0,3)$

Q $x^2-(a+1)x+2a=0$ has exactly one root in $[0,3]$ find range of a

$f(x)=x^2-(a+1)x+2a=0$

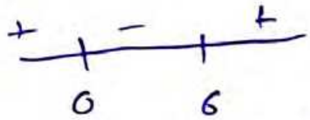
$f(0)\cdot f(3)<0$

$\Downarrow$

$\Rightarrow 2a(9-3a-3+2a)<0$

$\Rightarrow a(6-a)<0$

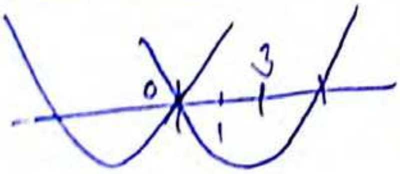
$\Rightarrow a(a-6)>0$



$a\in(-\infty,0)\cup(6,\infty)$ —(i)

Now two possibilities arise

(P_1)



$f(0)=0 \Rightarrow a=0$

eq^n: $x^2-x=0$

$x=0,1$

$a\neq 0$ rejected

~~rejected~~

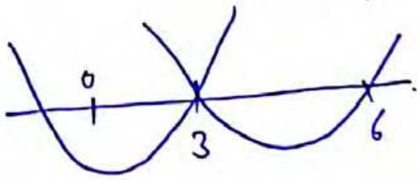
bcoz at $a=0$

$x=0,1$ and $[0,3]$

x have 2 roots

but ACTQ only one root lie b/w $[0,3]$

(P_2)



$f(3)=0 \Rightarrow 6-a=0 \Rightarrow a=6$

eq^n $x^2-7x+12=0$ → 3, β

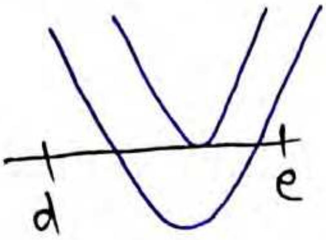
$3\cdot\beta=12$

$\beta=4$

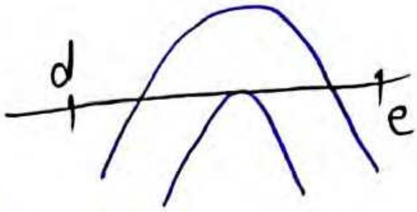
($a=6$ is accepted) —(ii)

$\therefore a\in(-\infty,0)\cup(6,\infty)\cup\{6\}$

$$\Rightarrow \boxed{a\in(-\infty,0)\cup[6,\infty)}$$



OR



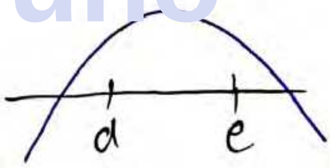
(1) $f(d) > 0$
(2) $f(e) > 0$
(3) $D \geq 0$
(4) $d < \frac{-b}{2a} < e$
(5) $a > 0$

(1) $f(d) < 0$
(2) $f(e) < 0$
(3) $D \geq 0$
(4) $d < \frac{-b}{2a} < e$
(5) $a < 0$

combine

(i) $D \geq 0$ (ii) $d < \frac{-b}{2a} < e$
(iii) $a \cdot f(d) > 0$ (iv) $a \cdot f(e) > 0$

Type V : One root is greater than e & other root is less than d ($d < e$



(1) $a > 0$
(2) $f(d) < 0$
(3) $f(e) < 0$
(4) $D > 0$

(1) $a < 0$
(2) $f(d) > 0$
(3) $f(e) > 0$
(4) $D > 0$

combine

(i) $a \cdot f(d) < 0$
(ii) $a \cdot f(e) < 0$
(iii) $D > 0$ (No Need)

$ax^2+bx+c=0$, where $a, b, c \in Q$ has rational roots if D is a perfect Square

Q Find no: of integral values α for which the quadratic eqn $x^2+\alpha x+\alpha+1=0$ has integral roots.

$x^2+\alpha x+\alpha+1=0$ we can assume α as integer clearly

all coeff are rational

$\therefore$ for roots be rational $\Rightarrow$ D should be perfect square.

$D = \alpha^2-4(\alpha+1) = \alpha^2-4\alpha-4 = m^2$, $m \in N$

$\Rightarrow \alpha^2-4\alpha-4 = m^2$

$\Rightarrow (\alpha-2)^2-8 = m^2$

$\Rightarrow (\alpha-2)^2-m^2 = 8$

$\Rightarrow \underset{A}{(\alpha-2-m)}\underset{B}{(\alpha-2+m)} = 8 \Rightarrow 1\times 8,\ 2\times 4,\ -8\times -1,\ -4\times -2$

m plus होता है तो B ~~a~~ A से बड़ा आएगा
इसलिए [illegible] cases की जरूरत नहीं

1×8 ✓ $\quad 8\times 1$ ⊗

$\alpha-2-m=1$	$\alpha-2-m=2$	$\alpha-2-m=-8$	$\alpha-2-m=-4$
$\alpha-2+m=8$	$\alpha-2+m=4$	$\alpha-2+m=-1$	$\alpha-2+m=-2$
$2\alpha-4=9$	$2\alpha-4=6$	$2\alpha-4=-9$	$2\alpha-4=-6$
$\alpha=13/2$ ×	$\alpha=5$ ✓	$\alpha=-5/2$ ×	$\alpha=-1$ ✓
(bcoz not an integer)		(bcoz not an integer)	

$\Rightarrow$ if $\alpha=-1$ or 5 root are rational

if $\alpha=-1$ eqn becomes $x^2-x=0 \Rightarrow x=0,1$

if $\alpha=5$ eqn becomes $x^2+5x+6=0 \Rightarrow x=-2,-3$

Both roots integer

Hence $\alpha=-1$ & $\alpha=5$ is valid for integral roots.

M-2

$$x^2+\alpha x+\alpha+1=0 \begin{matrix} \nearrow a \\ \searrow b \end{matrix} \text{ integral}$$

$$S = a+b = -\alpha$$

$$P = ab = \alpha+1$$

$$a+b+ab = 1$$

{ Simon's Factoring Technique }

$$\Rightarrow (a+1)+b(a+1)-1 = 0$$

$$\Rightarrow (a+1)(b+1) = 2$$

$$\Rightarrow a+1=1, \quad b+1=2 \quad\Rightarrow\quad a=0,\ b=1$$

OR

$$a+1=-2, \quad b+1=-2 \quad\Rightarrow\quad a=-2,\ b=-3$$

$$\alpha = -1, 5$$

$$\Rightarrow -\alpha = a+b \qquad\qquad -\alpha = -5$$

$$\alpha = -1 \qquad\qquad \alpha = 5$$

* condition for general two degree expression in x & y to be resolved as a product of two linear factors.

$$f(x,y) = ax^2+2hxy+by^2+2gx+2fy+c$$

$$= ax^2+(2hy+2g)x+by^2+2fy+c$$

{ let us find its roots. }

$$\therefore\ x = \frac{-(2hy+2g)\pm\sqrt{(2hy+2g)^2-4\cdot a\cdot(by^2+2fy+c)}}{2a}$$

$$= \frac{-(hy+g)\pm\sqrt{(h^2-ab)y^2+(2hg-2af)y+g^2-ac}}{a}$$

$$x = \frac{-(hy+g)+\sqrt{(h^2-ab)y^2+(2hg-2af)y+g^2-ac}}{a} = \alpha$$

$$x = \frac{-(hy+g)-\sqrt{(h^2-ab)y^2+(2hg-2af)y+g^2-ac}}{a} = \beta$$

$\therefore$ for linear factors

$px + qy + r$ $\downarrow$ linear in x & y

$(h^2 - ab)y^2 - y(2hg - 2af) + g^2 - ac$ should become a perfect square

$\Rightarrow \{D = 0\}$

$D = (2hg - 2af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$

$\frac{(\ )}{4}\Big(\Rightarrow 4(hg - af)^2 - (h^2 - ab)(g^2 - ac) = 0$

$\Rightarrow h^2g^2 + a^2f^2 - 2afhg - [h^2g^2 - h^2ac - g^2ab + a^2bc] = 0$

$\frac{(\ )}{a}\Big(\quad af^2 - 2fgh + ch^2 + bg^2 - abc = 0$

$\Rightarrow \boxed{abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$

$f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be resolved in to two linear factors if

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

OR

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Q: $2x^2 + 3xy + y^2 + 3x + 1 + 2y$ Factorize into linear factors

$f(x,y) = 2x^2 + 3xy + y^2 + 3x + 1 + 2y$

$a = 2$; $b = 1$; $2h = 3$, $h = 3/2$; $2g = 3$, $g = 3/2$; $2f = 2$, $f = 1$; $c = 1$

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 2\cdot1\cdot1 + 2\cdot1\cdot\frac{3}{2}\cdot\frac{3}{2} - 2\cdot1^2 - 1(3/2)^2 - 1\cdot(3/2)^2$

$= 2 + 9/2 - 2 - 9/4 - 9/4$

$= 0$

$\therefore$ given Expression can be factorized into 2 linear factor.

$F(x,y) = 2x^2 + 3xy + y$

Steps to Factorize

* Factorize the Homogenous part (degree same)

$$2x^2 + 3xy + y^2 = 2x^2 + 2xy + xy + y^2$$
$$= 2x(x+y) + y(x+y)$$
$$= (2x+y)(x+y)$$

* Add λ to first factor & μ to second factor & put their product equal to given expression,

$$(2x+y+\lambda)(x+y+\mu) = 2x^2 + 3xy + y^2 + 3x + 2y + 1$$

$2\mu x + \lambda x$

$\underbrace{(2\mu+\lambda)}_{\text{coeff. of } x} x$

$\mu y + \lambda y$

$\underbrace{(\mu+\lambda)}_{\text{coeff of } y} y$

coefficient of x : $2\mu + \lambda = 3$

coefficient of y : $\mu + \lambda = 2$

$\mu = 1, \lambda = 1$

$$\boxed{\therefore\ 2x^2 + 3xy + y^2 + 3x + 2y + 1 = (2x+y+1)(x+y+1)}$$

* A Golden Point :

for $y = f(x) = ax^2 + bx + c \quad (a \neq 0)$

if $f(p) < 0$ and $f(q) > 0$

i.e $f(p) \cdot f(q) < 0 \Rightarrow$ then the equation $ax^2 + bx + c = 0$ has exactly one root lying b/w p & q.